Student details

Name: ____________________

Mark: ____________________

2023

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Reference sheet is provided separately.
- Marks may be lost for poor working out and/or poor logic.

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Circle the BEST solution.

Section II Pages 6 – 14

90 marks

- Attempt Questions 11 – 34
- Your responses should include relevant mathematical reasoning and/or calculations.

Section I

10 marks
Attempt Questions 1 – 10

Circle the BEST solution below for Questions 1 – 10.

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1 Which of the following coordinates is in the 5th octant?

(A) (2, 3, –5)

(B) (–1, 8, 6)

(C) (8, –8, 7)

(D) (–4, –1, –9)

2 Consider the statement: $\forall a,b,c,d:\ a \cap b \Rightarrow c \cup d$.

Which of the following represents the statement's contrapositive?

(A) $\forall a,b,c,d:\ a \cap b \Leftarrow c \cup d$

(B) $\forall a,b,c,d:\ c \cap d \Leftarrow a \cup b$

(C) $\forall a,b,c,d:\ a \cap b \Rightarrow c \cap d$

(D) $\forall a,b,c,d:\ c \cap d \Rightarrow a \cup b$

3 Which of the following is equivalent to $2e^{\frac{2\pi}{3}i}$?

(A) $\sqrt{3} + i$

(B) $1 - \sqrt{3}i$

(C) $-\sqrt{3} - i$

(D) $-1 + \sqrt{3}i$

4 Which of the following equals to the magnitude of the vector $\begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\theta \end{pmatrix}$

(A) $\sec\theta$

(B) $\tan\left(\frac{\theta}{2}\right)$

(C) $\cot\theta$

(D) 1

5 Which of the following are the five roots of $z^5 - i = 0$?

(A) $z = \text{cis}\left(-\frac{7\pi}{10}\right),\ \text{cis}\left(-\frac{3\pi}{10}\right),\ \text{cis}\frac{\pi}{10},\ \text{cis}\frac{9\pi}{10},\ i$

(B) $z = \text{cis}\left(-\frac{9\pi}{10}\right),\ \text{cis}\left(-\frac{\pi}{10}\right),\ \text{cis}\frac{3\pi}{10},\ \text{cis}\frac{7\pi}{10},\ -i$

(C) $z = \text{cis}\left(-\frac{4\pi}{5}\right),\ \text{cis}\left(-\frac{2\pi}{5}\right),\ \text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ 1$

(D) $z = \text{cis}\left(-\frac{3\pi}{5}\right),\ \text{cis}\left(-\frac{\pi}{5}\right),\ \text{cis}\frac{\pi}{5},\ \text{cis}\frac{3\pi}{5},\ i$

6 What is the equivalent to $\int \frac{1}{\left(1+x^2\right)^2}\,dx$?

(A) $-\frac{1}{1+x^2}+c$

(B) $\frac{\sqrt{1+x^2}}{2}+c$

(C) $\frac{x}{2\left(1+x^2\right)}+\frac{\tan^{-1}x}{2}+c$

(D) $\ln\left|\frac{x+\sqrt{1+x^2}}{2}\right|+c$

7 Which of the following represents the Cartesian equation of the vector passing through the point (4, 3, –1) and is parallel to the vector $2i - 4j + 3k$?

(A) $(x-4)^2+(y-3)^2+(z+1)^2=29$

(B) $(x-2)^2+(y+4)^2+(z-3)^2=26$

(C) $\frac{x-4}{2}=\frac{3-y}{4}=\frac{z+1}{3}$

(D) $\frac{x-2}{4}=-\frac{y+4}{3}=z-3$

8 Which of the following statements is **<u>FALSE</u>**?

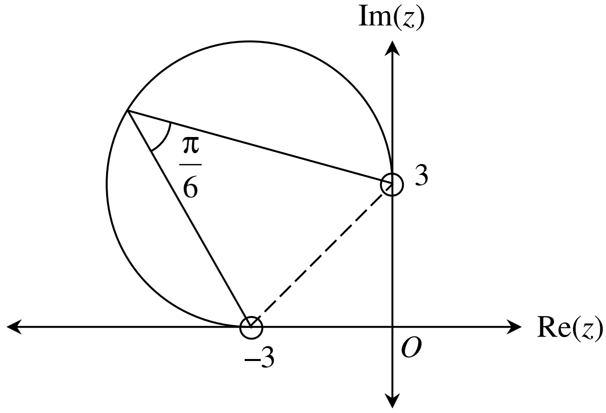
(A) $\forall p\in\mathbb{R},\exists q\in\mathbb{R}:\quad p^4=2q^3$

(B) $\forall p\in\mathbb{R},\exists q\in\mathbb{R}:\quad p=q \iff \cos p=\cos q$

(C) $\forall p\in\mathbb{R},\exists q\in\mathbb{R}:\quad \sqrt{p}<e^q$

(D) $\forall p\in\mathbb{R},\exists q\in\mathbb{R}:\quad |p|>\frac{1}{q}$

9 Consider the following diagram:



The diagram shows the locus of which equation?

(A) $\arg\left(\frac{z+3}{z-3i}\right)=\frac{\pi}{6}$

(B) $\arg\left(\frac{z-3i}{z+3}\right)=\frac{\pi}{6}$

(C) $\arg\left[(z+3)(z-3i)\right]=\frac{\pi}{6}$

(D) $\left|z^2-9\right|=\frac{\pi}{6}$

10 The tides observed in a harbour can be modelled using simple harmonic motion. On average, the first daily low tide is observed at 6:45am while the first high tide is observed at 1:30pm. Low tides are measured at 8m above sea level on average while high tides are measured at 14m above sea level.

Which of the following equations can be used to model the height of the tides (h) where t represents time in hours after the first daily low tide?

(A) $h=22+3\cos\left(\frac{4\pi}{27}t\right)$

(B) $h=22+3\cos\left(\frac{27\pi}{4}t\right)$

(C) $h=11-3\sin\left(\frac{27\pi}{4}t\right)$

(D) $h=11-3\cos\left(\frac{4\pi}{27}t\right)$

Section II

90 marks
Attempt Questions 11–35

In Questions 11–35, your responses should include relevant mathematical reasoning and/or calculations.

Question 11

If $z = -4 + 3i$, express each of the following in the form $a + ib$, where a and b are real.

(a) $z + \overline{z}$. **1**

(b) $2iz$. **1**

(c) $\dfrac{2}{z}$. **1**

Question 12

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

(a) Find the value of $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$. **2**

(b) Find the size of the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (nearest degree). **1**

(c) Find the vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. **1**

Question 13

Prove by contradiction that $\sqrt{5}$ is irrational. **3**

Question 14

Find $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$. **2**

Question 15

Find the square roots of $-15-8i$. **2**

Question 16

a, b and c are the three sides of a triangle.

(a) Show that $ab+ac+bc \le a^2+b^2+c^2$. **1**

(b) Hence, or otherwise, show that $3(ab+ac+bc) \le (a+b+c)^2 \le 4(ab+ac+bc)$. **3**

Question 17

(a) Find real numbers a, b and c such that $\dfrac{10x^2-23x+8}{(x^2+3)(x-4)} = \dfrac{a}{x-4} + \dfrac{bx+c}{x^2+3}$. **2**

(b) Hence, or otherwise, find $\displaystyle\int \frac{10x^2-23x+8}{(x^2+3)(x-4)}\,dx$. **2**

Question 18

Consider the statement $\forall a \in \mathbb{Z}^+, \exists b \in \mathbb{Z}^+, b = a-1$.

(a) Write down the negation of the statement. **1**

(b) Prove the statement is false by using a suitable counterexample. **2**

Question 19

Consider the following vector equations of two lines: **2**

$$\underset{\sim}{r}_1 = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} 2 \\ -6 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}.$$

Determine whether the two lines have a point of intersection or if they are skew. Show all working.

Question 20

Use integration by parts to find $\int x \tan^{-1} x \, dx$. **3**

Question 21

A particle is moving along a straight line where its displacement x metres from O after t seconds is given by the formula:

$$x = 3\sin 2t + 3\sqrt{3}\cos 2t.$$

(a) Show that the particle moves with simple harmonic motion. **1**

(b) Find an expression for v^2 in terms of x, where v is the velocity of the particle. **2**

Question 22

On an Argand diagram, shade the region where: **2**

$$|z - \overline{z}| \le 4 \quad \text{and} \quad -\frac{\pi}{4} \le \text{Arg}(z) \le \frac{\pi}{2}.$$

Question 23

Consider the sphere S: $x^2 + y^2 + z^2 = 36$ with a centre at O and the vector L: $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

(a) The sphere S and the vector L intersect at the points P and Q. Find the coordinates of P and Q. **2**

(b) Find the size of $\angle POQ$, rounding your solution to the nearest degree. **2**

Question 24

(a) Using De Moivre's theorem, show that: $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. **2**

(b) Hence, or otherwise, show that the roots of $16x^4 - 16x^2 + 1 = 0$ has roots $x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}$ and $\cos\frac{11\pi}{12}$. **2**

(c) Hence, or otherwise, show that $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$. **2**

Question 25

By using an appropriate trigonometric substitution, find $\int \sqrt{x^2 - 4}\, dx$. **3**

Question 26

A heavy ball of mass m kg is dropped from rest in a medium with resistance of $\frac{1}{40}mv^2$, where the velocity of the ball is v m/s. After t seconds the ball has fallen x metres.

Assuming gravity is 10 m/s^2,

(a) Show that the ball's acceleration a is given by $a = \frac{1}{40}\left(400 - v^2\right)$. **1**

(b) Show that $v = 20\left(1 - \frac{2}{1 + e^t}\right)$. **3**

(c) Show that $x = 20\left(t + 2\log_e\left(\frac{1 + e^{-t}}{2}\right)\right)$. **2**

Question 27

Consider the function $y = x^2e^x$. Prove by mathematical induction for $n \in \mathbb{Z}^+$: **3**

$$\frac{d^n}{dx^n}x^2e^x = \left(n\left(n-1\right) + 2nx + x^2\right)e^x.$$

Question 28

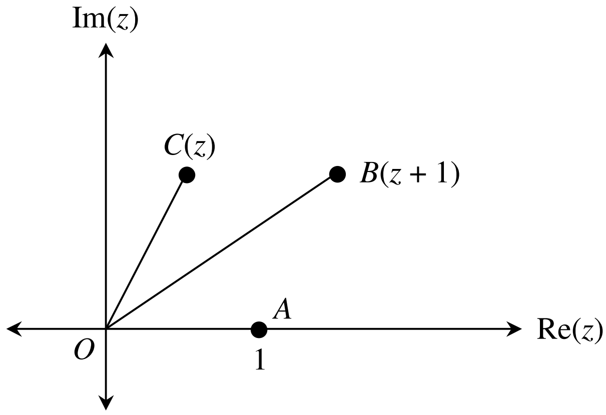
Let $I_n = \int \frac{x^n}{\sqrt{a^2 - x^2}}\,dx$ for $n \in \mathbb{Z}^+$.

(a) Show that $nI_n = -x^{n-1}\sqrt{a^2 - x^2} + a^2\left(n-1\right)I_{n-2}$ for integers $n \geq 2$. **3**

(b) Hence, find $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$. **1**

Question 29

In the diagram below, the points O, A, B and C represent the complex numbers 0, 1, $z + 1$ and z respectively, where $z = \cos\theta + i\sin\theta$ and $0 < \theta < \pi$.



(a) Explain why $OABC$ is a rhombus. **1**

(b) Show that $\dfrac{z-1}{z+1}$ is purely imaginary. **2**

(c) In terms of θ, find the modulus and argument of $z + 1$. **2**

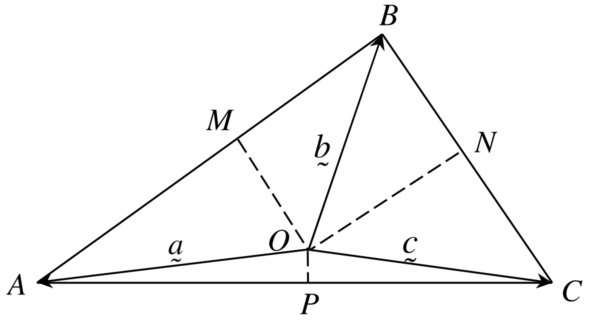
Question 30

Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two vectors such that $\underset{\sim}{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. **2**

If $a + b + c > 9$ and $a, b, c \in \mathbb{R}$, prove: $a^2 + b^2 + c^2 > 27$.

Question 31

In ΔABC, points M, N and P are the midpoints of AB, BC and AC respectively, where ON and OP are the perpendicular bisectors of BC and AC respectively. The vertices A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively, as shown in the diagram below.



Using vectors,

(a) Show that $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$. **2**

(b) Show that OM is the perpendicular bisector of AB. **2**

(c) Hence, or otherwise, show that the three perpendicular bisectors of the sides of ΔABC are concurrent. **1**

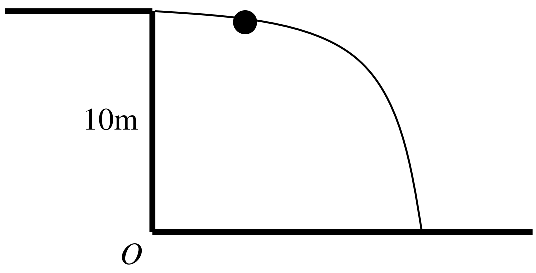
Question 32

Find the values of θ for the following equation, where $-\pi \leq \theta \leq \pi$: **3**

$$\left|e^{4i\theta} + 1\right| = \sqrt{3}$$

Question 33

An object of mass 1 kg was launched horizontally off a 10m high cliff with initial velocity of 1 metre per second. It experiences air resistance of $0.4v^2$ and gravity of 10 m/s^2, as shown in the diagram below.



The horizontal and vertical equations of the object's acceleration is as follows (**DO NOT PROVE THESE**):

$$\ddot{x} = -\frac{2\dot{x}^2}{5} \qquad \ddot{y} = -10 - \frac{2\dot{y}^2}{5}$$

(a) Prove the following horizontal and vertical equations of motion. **4**

$$\dot{x} = \frac{5}{2t + 5} \qquad \dot{y} = -5\tan(2t)$$

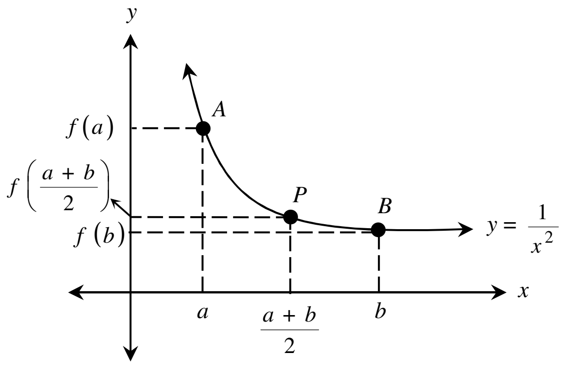
$$x = \frac{5}{2}\log_e\left(\frac{2t + 5}{5}\right) \qquad y = \frac{5}{2}\log_e\left|\cos(2t)\right| + 10$$

(b) Hence, or otherwise, find how long it takes for the object to hit the ground, rounding your solution to one decimal place. **1**

(c) Consider the situation where the object experiences no air resistance. Comparing this to the situation in parts (a) and (b), find be the difference in horizontal distance when the object lands on the ground. Round your solution to one decimal place. **2**

Question 34

Two points A and B lie on the curve $y = \frac{1}{x^2}$ at $x = a$ and $x = b$ respectively. Point P also lies on the curve at $x = \frac{a+b}{2}$, as shown in the diagram below.



By comparing specific area in the diagram above, it can be shown that:

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2}$$ [**DO NOT PROVE THIS**].

(a) If $a = n - 1$ and $b = n$, where $a, b \in \mathbb{Z}^+$ and $1 < a < b$, show that: **2**

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right).$$

(b) Show that: $4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right).$ **2**

(c) Show that: $\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ **1**

(d) Hence, show that: $\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}.$ **2**

End of paper.