NEX

Student details

Name: ____________________

Mark: ____________________

2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- Reference sheet is provided separately.
- Marks may be lost for poor working out and/or poor logic.

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Circle the BEST solution.

Section II Pages 6 – 12

90 marks

- Attempt Questions 11 – 34
- Your responses should include relevant mathematical reasoning and/or calculations.

Section I

10 marks
Attempt Questions 1 – 10

<u>Circle the BEST solution</u> below for Questions 1 – 10.

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1 Which of the following coordinates is in the 6^{th} octant?

(A) (4, –2, 1)

(B) (8, 7, –3)

(C) (–2, –2, –5)

(D) (–1, 1, –9)

2 Which of the following is parallel to vector $\underset{\sim}{r} = \begin{pmatrix} 8 \\ 9 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$, where λ is a constant.

(A) Vector passing through $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

(B) The vector $\underset{\sim}{a} = \begin{pmatrix} 8 \\ 9 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

(C) Vector passing through $\begin{pmatrix} 1 \\ 3 \\ -9 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 12 \\ 5 \end{pmatrix}$.

(D) The vector $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix}$.

3 Consider the statement: $\forall P,Q,R\colon \{P \Rightarrow R\} \Rightarrow \{P \Rightarrow Q \text{ OR } Q \Rightarrow R\}$.

Which of the following represents the statement's contrapositive?

(A) $\forall P,Q,R\colon \neg\{P \Rightarrow R\} \Rightarrow \neg\{P \Rightarrow Q \text{ OR } Q \Rightarrow R\}$

(B) $\forall P,Q,R\colon \neg\{P \Rightarrow R\} \Rightarrow \neg\{P \Rightarrow Q \text{ AND } Q \Rightarrow R\}$

(C) $\forall P,Q,R\colon \neg\{P \Rightarrow Q \text{ AND } Q \Rightarrow R\} \Rightarrow \neg\{P \Rightarrow R\}$

(D) $\forall P,Q,R\colon \neg\{P \Rightarrow Q \text{ OR } Q \Rightarrow R\} \Rightarrow \neg\{P \Rightarrow R\}$

4 Which of the following are the solutions to the equation $z^2 - 2\cos\theta z + 1 = 0$?

(A) $\cos\theta \pm i\sin\theta$

(B) $\sin\theta \pm i\cos\theta$

(C) $\sqrt{3}\cos\theta \pm i\sin\theta$

(D) $\sec\theta \pm i\cot\theta$

5 What is the equivalent to $\int \frac{1}{e^x+1}\,dx$?

(A) $\frac{e^x}{\left(e^x+1\right)^2} + c$

(B) $\frac{\sqrt{e^x+1}}{2} + c$

(C) $e^x\left(e^x+1\right)^2 + c$

(D) $\ln\left|\frac{e^x}{e^x+1}\right| + c$

6 ω is a complex cube root of unit, where $\omega \neq 1$.

Which of the following equates to $(1 + 3\omega + \omega^2)(1 + \omega - 3\omega^2)$?

(A) -8

(B) 3

(C) 12

(D) 16

7 Which of the following is the best statement regarding $\int \sec x \; dx$?

(A) It is equivalent to $\log_e \left|\sec x + \tan x\right| + c$.

(b) It is equivalent to $\log_e \left|\dfrac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right| + c$.

(C) All of the above.

(D) None of the above.

8 A cannonball weighing m kilograms was fired from the ground with initial velocity of u metres per second at an angle of θ to the horizon. In addition to gravity of g m/s^2, the cannonball experiences air resistance proportional to the cannonball's velocity v of mkv.

Which of the following represents the time the cannonball takes to reach its maximum vertical height?

(A) $\dfrac{1}{k}\log_e\left(\dfrac{g + ku\sin\theta}{g}\right)$

(B) $\left(g + ku\sin\theta\right)e^{-k}$

(C) $\dfrac{1}{k}\log_e\left(\cos\left(\sqrt{gk}\right) + \dfrac{ku\sin\theta}{\sqrt{gk}}\sin\left(\sqrt{gk}\right)\right)$

(D) $\left(\cos\left(\sqrt{gk}\right) + \dfrac{ku\sin\theta}{\sqrt{gk}}\sin\left(\sqrt{gk}\right)\right)e^{-k}$

9 What is the exact value of $(-1)^i$?

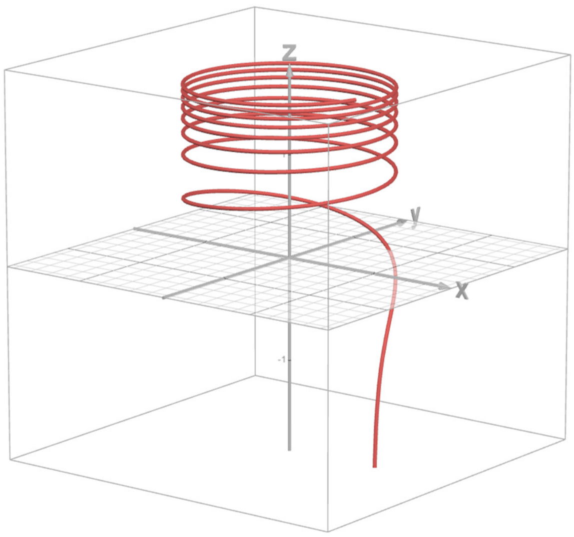
(A) e

(B) e^{π}

(C) $e^{-\frac{\pi}{2}}$

(D) $e^{-\pi}$

10 Consider the curve in the following diagram:



Which of the following position vectors is best represented by the given curve?

(A) $\underset{\sim}{r} = \left(\cos t, \sin t, \frac{1}{t}\right)$

(B) $\underset{\sim}{r} = \left(\cos t, \sin t, \log_e t\right)$

(C) $\underset{\sim}{r} = \left(\cos t, \sin t, t^2\right)$

(D) $\underset{\sim}{r} = \left(\cos t, \sin t, e^{-t}\right)$

Section II

90 marks
Attempt Questions 11–35

In Questions 11–35, your responses should include relevant mathematical reasoning and/or calculations.

Question 11

If $z = 2 - 5i$ and $w = -2 + 3i$, express each of the following in the form $a + ib$, where $a, b \in \mathbb{R}$.

(a) $z - \overline{z}$. **1**

(b) w^2. **1**

(c) $\dfrac{10}{w}$. **1**

(d) $\sqrt{z + w}$. **2**

Question 12

Consider the statement: "If I were a nerd then I would be rich and have no friends". **2**

Write down the contrapositive of the statement.

Question 13

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$.

(a) Find $|\underset{\sim}{a}|$. **1**

(b) Find the size of the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (nearest degree). **1**

(c) Find the vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. **1**

Question 14

Find $\int \frac{1}{x(\ln x)^3} \, dx$. **2**

Question 15

A particle moving along a straight line with displacement of x metres and velocity v ms^{-1} moves according to the formula:

$$v^2 = 45 - 36x - 9x^2.$$

(a) Show that the particle moves with simple harmonic motion. **1**

(b) Find the particle's amplitude. **2**

Question 16

Consider two complex numbers $z = 1 - i$ and w such that $|zw| = \sqrt{8}$ and $\arg(zw) = \frac{\pi}{12}$.

(a) Express z in the form $re^{i\theta}$ where $r > 0$ and $-\pi \le \theta \le \pi$. **1**

(b) Find w in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

Question 17

Given that $z = x + iy$ is a point on the Argand plane such that $z\overline{z} - 2(z + \overline{z}) = 21$.

(a) Find the locus of z. **2**

(b) Hence, or otherwise, determine the maximum value of $|z - 4|$. **2**

Question 18

An object initially on the ground was projected diagonally into the air. After t seconds, its velocity vector is given by the equation:

$$\underset{\sim}{v} = 20\sqrt{3}e^{-0.04t}\underset{\sim}{i} + \left(270e^{-0.04t} - 250\right)\underset{\sim}{j}.$$

where velocity is measured in metres per second.

(a) Show that the time it takes to reach its maximum height is $25\log_e\left(\frac{27}{25}\right)$. **1**

(b) Hence, or otherwise, find the maximum height attained by the object, rounding your solution to the nearest metre. **2**

Question 19

Consider the following vector equations of two lines: **2**

$$\underset{\sim}{r}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

Determine whether the two lines have a point of intersection or are skew. Show all working.

Question 20

(a) Find real numbers a and b such that: $\dfrac{3x^3 + 10x^2 + 21x + 78}{(x^2+9)(x+2)} = 3 + \dfrac{a}{x+2} + \dfrac{b}{x^2+9}$. **2**

(b) Hence, or otherwise, find $\displaystyle\int \frac{3x^3 + 10x^2 + 21x + 78}{(x^2+9)(x+2)}\,dx$. **2**

Question 21

Solve for z: $z^2 = |z|^2 - 8.$ **2**

Question 22

On an Argand diagram, shade the region where: **2**

$$|z - 1| \leq 4 \quad \text{and} \quad |z + 2i| \leq 9 .$$

Question 23

Use integration by parts to find $\int e^{3x} \sin 4x \, dx$. **3**

Question 24

Find a vector that is perpendicular to the vectors $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. **3**

Question 25

(a) If $z = \cos\theta + i\sin\theta$, show that: $z^n + z^{-n} = 2\cos(n\theta)$. **2**

(b) Hence, solve for z: $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$. **2**

Question 26

Find $\int \sqrt{\frac{8 - x}{x}} \, dx$. **3**

Question 27

z_1 and z_2 are two complex numbers. Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. **2**

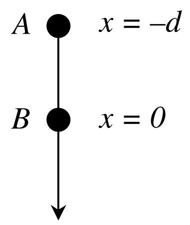
Question 28

Prove by contradiction that there are no rational solutions to the equation: **3**

$$z^3 + 3z + 3 = 0.$$

Question 29

Two objects, A and B, with equal mass m kg were released vertically downwards through a medium with resistance mkv, where the velocity of the object is v m/s and k is a constant. Object A is released with initial velocity U m/s from a point d metres above the object B, which was released from rest from the origin.



Assume gravity is g m/s^2.

(a) For object A,

(i) Show that the object's acceleration a is given by $a = g - kv$. **1**

(ii) Show that $v_A = \frac{g}{k} - \left(\frac{g - kU}{k}\right)e^{-kt}$. **2**

(iii) Show that $x_A = \frac{gt - kd}{k} + \left(\frac{g - kU}{k^2}\right)\left(e^{-kt} - 1\right)$. **2**

(b) For object B, using the expressions above, write down similar expressions for velocity v_B and displacement x_B. **2**

(c) Find when the objects collide. **2**

Question 30

Evaluate: $\int_0^{\frac{\pi}{4}} \sec^3 x \, dx$. **3**

Question 31

(a) Show that: $\cot 2x - \tan 2x = 2\cot 4x$. **2**

(b) Hence, or otherwise, prove by mathematical induction for $n \in \mathbb{Z}^+$: **3**

$$\tan x + 2\tan 2x + 4\tan 4x + \ldots + 2^{n-1} \tan\left(2^{n-1} x\right) = \cot x - 2^n \cot\left(2^n x\right)$$

Question 32

Find $\int \frac{1}{\sqrt{e^{2x} + 1}} \, dx$ [Hint: Apply the substitution method for e^x]. **3**

Question 33

If $a, b, c \in \mathbb{R}$ and $a > b > c$, prove: $|a - b| + |c - b| \geq a - c$. **2**

Question 34

Consider the function $f(x) = 2\log_e x - \frac{x^2 - 1}{x}$, $x > 0$.

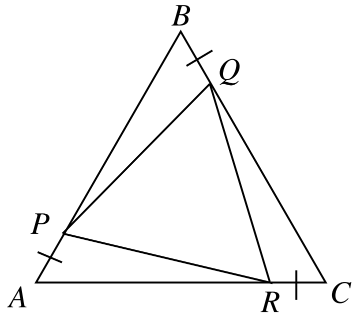
(a) Show that the only root of $f(x)$ is at $x = 1$. **2**

(b) Let $g(x) = \frac{x\log_e x}{x^2 - 1}$, $x > 0$ and $x \neq 1$. **3**

Show that $0 < g(x) < \frac{1}{2}$ for all $x > 0$ and $x \neq 1$.

Question 35

ΔABC is an equilateral triangle with points P, Q and R lying on AB, BC and AC respectively such that $AP = BQ = CR$, as shown in the diagram below.



Let $\hat{\underset{\sim}{a}}$, $\hat{\underset{\sim}{b}}$ and $\hat{\underset{\sim}{c}}$ be the unit vectors in the direct of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively, and let $\overrightarrow{AB} = \lambda \hat{\underset{\sim}{a}}$ and $\overrightarrow{AP} = \mu \hat{\underset{\sim}{a}}$.

(a) (i) In terms of $\hat{\underset{\sim}{a}}$, $\hat{\underset{\sim}{b}}$, λ and μ, find the expressions for $\overrightarrow{BC}$ and $\overrightarrow{BQ}$. **1**

(ii) Show that $\left|\overrightarrow{AQ}\right| = \sqrt{\lambda^2 + \lambda\mu + \mu^2}$. **2**

(b) Hence, or otherwise, show that ΔPQR is an equilateral triangle. **1**

Question 36

Let $I_n = \int_0^p x^n \sqrt{p^2 - x^2}\, dx$ for $n \in \mathbb{Z}^+$ and $p \in \mathbb{R}^+$.

(a) Show that $I_n = p^2 \dfrac{n-1}{n+2} I_{n-2}$ for integers $n \geq 2$. **3**

(b) Show that $I_{2n} = \dfrac{\pi p^{2n+2}(2n)!}{2^{2n+2} n!(n+1)!}$ for integers $n \geq 2$. **3**

End of paper.