NExT EDUCATION

2019

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 14

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10

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1 Which of the following is equivalent to i^{19997}?

(A) 1

(B) i

(C) $-i$

(D) -1

2 Find the value of the eccentricity (e) of the following equation: $0.2x^2 + 0.25y^2 = 1$

(A) $e = 0.3$

(B) $e = \dfrac{1}{\sqrt{5}}$

(C) $e = \dfrac{4}{5}$

(D) $e = \dfrac{3}{2}$

3 If $z = 1 + i\sqrt{3}$, which of the following is equivalent to z^{24}?

(A) 2^{12}

(B) 2^{24}

(C) $2^{12}\sqrt{3}$

(D) $2^{24}\sqrt{3}$

4 The roots of the equation $x^3 + 2x - 8 = 0$ are α, β and γ.

Which of the following equations has roots of $(1 - \alpha)$, $(1 - \beta)$ and $(1 - \gamma)$?

(A) $x^3 - 3x^2 + 5x + 5 = 0$

(B) $2x^3 + x^2 - 4x + 2 = 0$

(C) $5x^3 + 2x^2 + 3x - 1 = 0$

(D) $x^3 - 2x - 1 = 0$

5 The derivative $\frac{dy}{dx}$ of the curve $y^3 = x^2 + xy$ is:

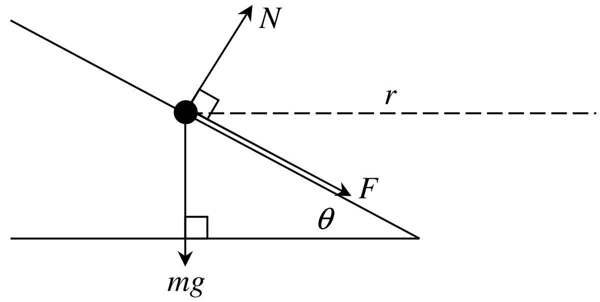
(A) $\frac{dy}{dx} = \frac{2x - y}{3y^2 + y}$

(B) $\frac{dy}{dx} = \frac{2x}{3y^2 + y}$

(C) $\frac{dy}{dx} = \frac{3y^2 - 2x}{x}$

(D) $\frac{dy}{dx} = \frac{2x + y}{3y^2 - x}$

6 A vehicle of mass m kilograms moving with velocity v metres per second is rounding a bend with radius r metres banked at an angle of θ. A lateral (sideways) force F is acting between its tyres and the road, and a normal reaction force N is acting on the tyres, as shown in the diagram.



By resolving forces, which of the following is true for F?

(A) $F = m\left(g\cos\theta - \dfrac{v^2}{r}\sin\theta\right)$

(B) $F = m\left(\dfrac{v^2}{r}\cos\theta - g\sin\theta\right)$

(C) $F = m\left(\dfrac{v^2}{r}\sin\theta + g\cos\theta\right)$

(D) $F = m\left(\dfrac{v^2}{r} - g\sin 2\theta\right)$

7 A committee of three is to be selected at random from six women and n men, where n is a positive integer. What is the number of possible committees containing exactly one woman?

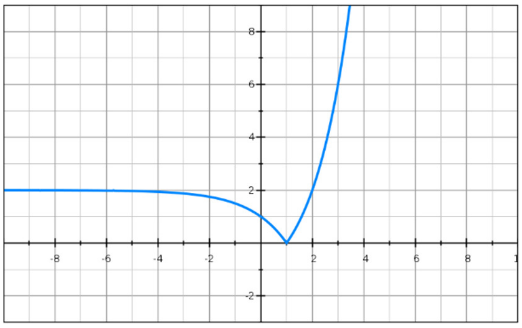
(A) ${}^{n}C_{2}$

(B) 20

(C) $6n$

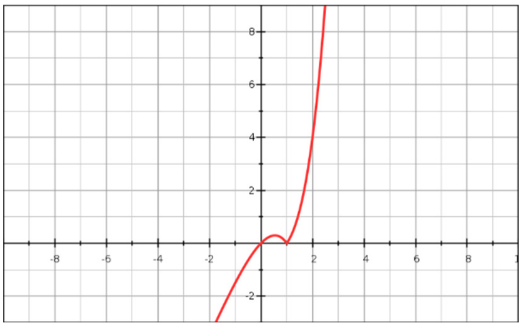
(D) $3n(n-1)$

8 The following depicts the function $y = f(x)$:

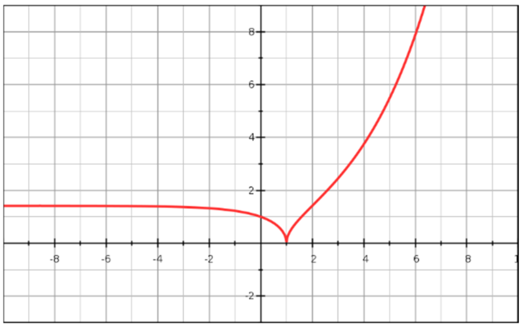


Which of the following graphs best represents $y = xf(x)$?

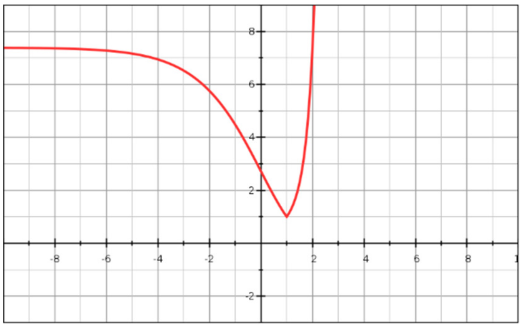
(A)



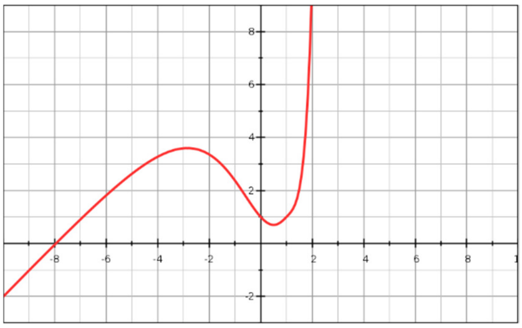
(B)



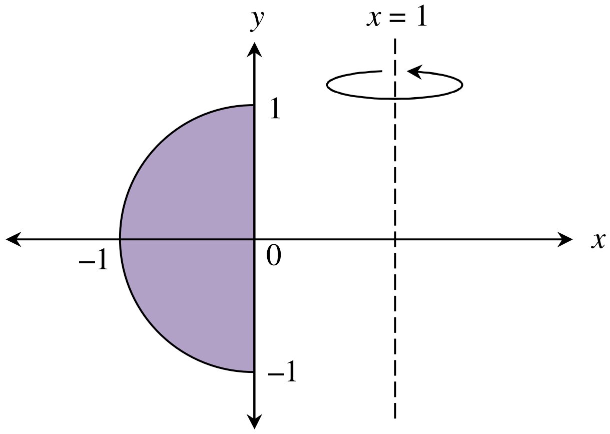
(C)



(D)



9 The region bound by the circle $x^2 + y^2 = 1$ for $-1 \le x \le 0$ is rotated about the line $x = 1$, as shown in the diagram below.



By using the method of cylindrical shells, which of the following expressions represent the volume of the solid formed?

(A) $2\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$

(B) $2\pi\int_{-1}^{0}(1+x)\sqrt{1-x^2}\,dx$

(C) $4\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$

(D) $4\pi\int_{-1}^{0}(1+x)\sqrt{1-x^2}\,dx$

10 If w is a complex cube root of unity, $w \neq 1$, which of the following is equivalent to the expression $\frac{1}{1+w}+\frac{1}{1+w^2}$?

(A) -1

(B) 0

(C) 1

(D) 2

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question on a NEW page on your OWN PAPER.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a NEW page on your OWN PAPER.

(a) If $z = -3 + 4i$, express each of the following in the form $a + ib$, where a and b are real.

(i) $3\overline{z}$. **1**

(ii) $z\overline{z}$. **1**

(iii) z^2. **1**

(iv) $\dfrac{1}{z}$. **1**

(v) $\sqrt{z}$. **2**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{8x - 4x^2}}$. **2**

(c) (i) Find real numbers a, b and c such that $\dfrac{x^2 + 2}{x^2 - x - 2} = a + \dfrac{b}{x - 2} + \dfrac{c}{x + 1}$. **1**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{x^2 + 2}{x^2 - x - 2}\, dx$. **3**

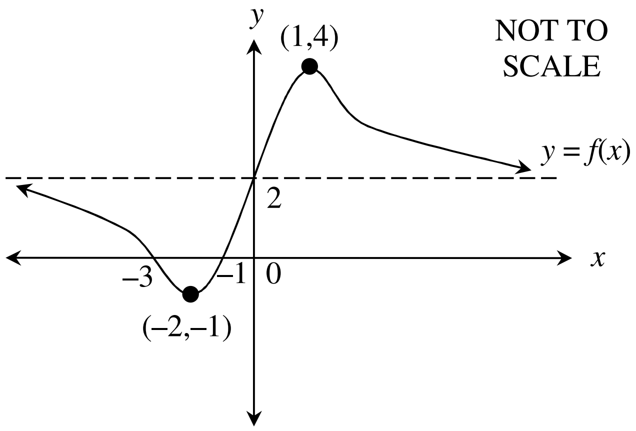
(d) Evaluate $\displaystyle\int_1^{e^2} x^2 \log_e x \, dx$. **3**

End of Question 11.

Question 12 (15 marks) Use a NEW page on your OWN PAPER.

(a) Evaluate $\int_0^{\frac{\pi}{3}} \sec^4\theta \tan\theta \, d\theta$. **3**

(b) The diagram shows the graph of a function $y = f(x)$, where there are stationary points at (1,4) and (–2,–1), x-intercepts at $x = -1$ and $x = -3$, and an oblique asymptote at $y = 2$.



Sketch the following curves on separate half-page diagrams.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \cos^{-1}\left[f(x)\right]$ **2**

(c) If $ax^4 + bx^3 + dx + e = 0$ has a non-zero triple root, show that: $4a^2d + b^3 = 0$. **3**

(d) If $z = \cos\theta + i\sin\theta$, show that $\dfrac{1}{1+z} = \dfrac{1}{2}\left(1 - i\tan\dfrac{\theta}{2}\right)$. **3**

End of Question 12.

Question 13 (15 marks) Use a NEW page on your OWN PAPER.

(a) If a, b and c are positive,

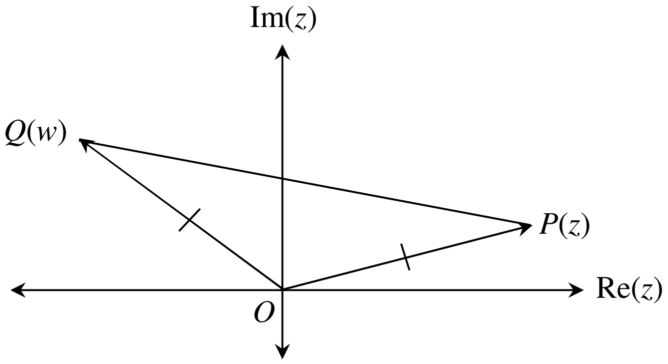
(i) Prove that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence, or otherwise, prove that $a^3 + b^3 \geq ab(a + b)$. **2**

(iii) Hence, or otherwise, prove that $(a + b)(b + c)(c + a) \geq 8abc$. **2**

(b) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4\sin x - 2\cos x + 6}\,dx$. **4**

(c) The complex numbers z and w are represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively, as shown in the diagram below.



Given that ΔPOQ is isosceles and $\angle POQ = \frac{2\pi}{3}$,

(i) Find an expression for w in terms of z. **1**

(ii) Hence, show that $(z + w)^2 = zw$. **3**

(d) α, β and γ are the roots of the equation $x^3 - 2x^2 - 5x - 1 = 0$. Find an equation with roots of $\frac{1}{\sqrt{\alpha}}$, $\frac{1}{\sqrt{\beta}}$ and $\frac{1}{\sqrt{\gamma}}$. **2**

End of Question 13.

Question 14 (15 marks) Use a NEW page on your OWN PAPER.

(a) (i) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$ for integers $n \geq 0$. **4**

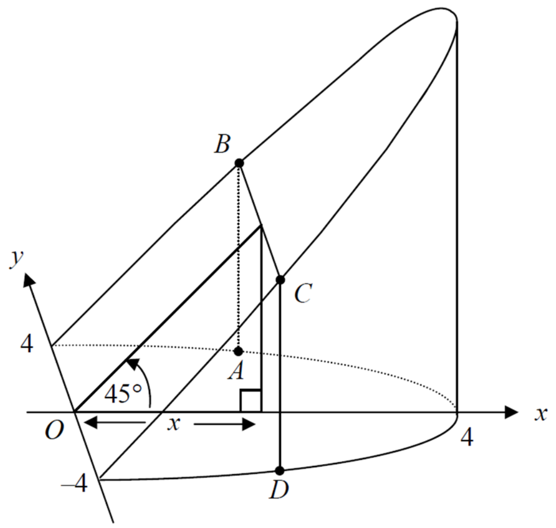
Show that $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$ for integers $n \geq 1$.

(ii) Hence, find the value of $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx$. **2**

(b) (i) On an Argand diagram, sketch the locus of the point P representing the complex number z such that $\left|z - \left(\sqrt{3} + i\right)\right| = 1$. **2**

(ii) Find the set of possible values of $|z|$ and $\text{Arg}(z)$. **2**

(c) A wedge was created by cutting a right cylinder of radius 4 units at 45° through a diameter of its base, as shown in the diagram below.



The wedge comprises of rectangular cross-sections taken perpendicular to the base of the wedge at a distance of x from the y-axis.

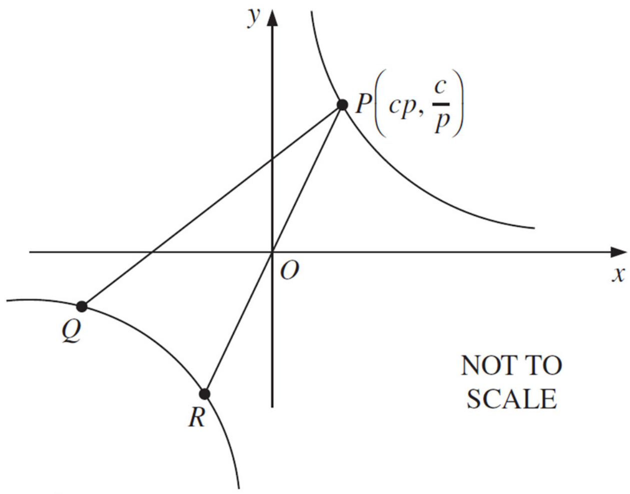
(i) Show that the area of $ABCD$ is given by $2x\sqrt{16 - x^2}$. **2**

(ii) Hence, find the exact volume of the wedge. **3**

End of Question 14.

Question 15 (15 marks) Use a NEW page on your OWN PAPER.

(a)



The point $P\left(cp, \frac{c}{p}\right)$, where $p \neq \pm 1$, is a point on the hyperbola $xy = c^2$, and the normal to the hyperbola at P intersects the second branch at Q. The line through P and the origin O intersects the second branch at R.

Given that the equation of the normal at P is $py - c = p^3(x - cp)$,

(i) Show that the x-coordinate of P and Q satisfy the equation **2**

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0 .$$

(ii) Find the coordinates of Q, and show that $\angle QRP$ is a right angle. **3**

(b) Let $w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$,

(i) Show that w^k is a solution of $z^7 - 1 = 0$, where k is an integer. **1**

(ii) Show that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$. **2**

(iii) Hence, or otherwise, show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$. **2**

Question 15 continues on the next page.

(c) An object of mass m kg is dropped from rest from the top of a cliff 30 metres high. The resistance to its motion has magnitude $\frac{1}{20}mv^2$ when the velocity of the object is v m/s. The object has fallen x metres after t seconds.

(i) Show that the object's terminal velocity V is $\sqrt{20g}$. **1**

(ii) Find the expression for v in terms of x. **3**

(iii) Find the percentage of its terminal velocity that the object will attain just prior to hitting the ground. **1**

End of Question 15.

Question 16 (15 marks) Use a NEW page on your OWN PAPER.

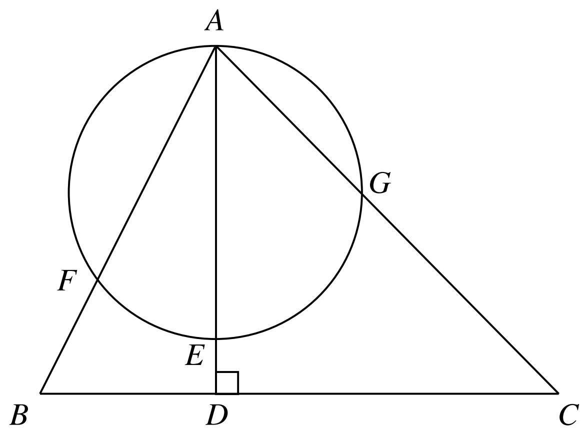
(a) (i) Using the principles of mathematical induction, prove that for $n > 1$ and $x > -1$: **3**

$$(1 + x)^n > 1 + nx$$

(ii) Hence, deduce that $\left(1 - \dfrac{1}{2n}\right)^n > \dfrac{1}{2}$ for $n > 1$. **1**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{x+1} + x + 1}$. **3**

(c) In triangle ABC, AD is a perpendicular drawn to BC. Points E, F and G lie on the circumference of a circle and also lie on the lines AB, AD and AC respectively. AE is the diameter of the circle. This is shown in the diagram below.

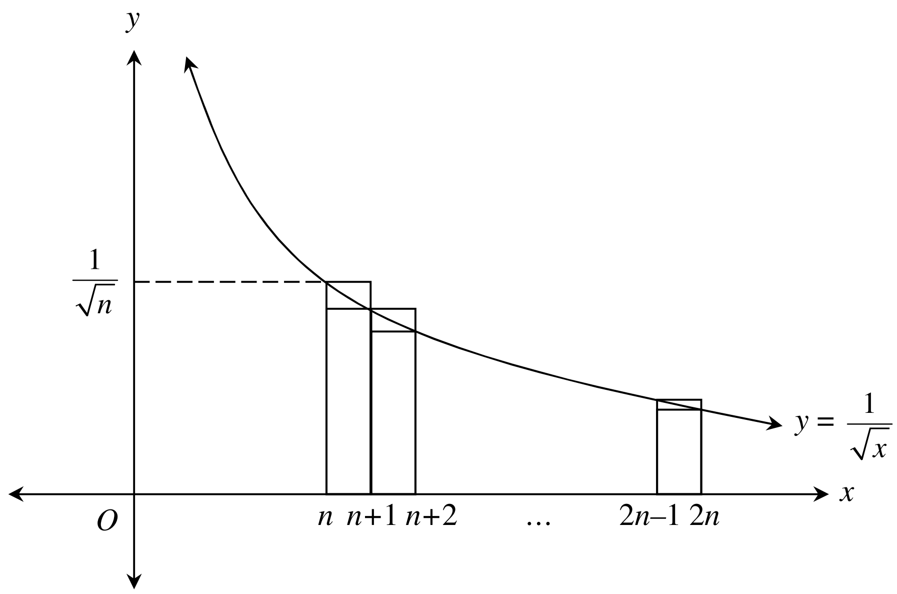


Prove that the $BFGC$ is a cyclic quadrilateral. **3**

Question 16 continues on the next page.

(d) (i) Show that $\int_n^{2n} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{n}\left(\sqrt{2}-1\right)$. **1**

(ii) In the diagram below, the graph of $y = \frac{1}{\sqrt{x}}$ is drawn and n upper and lower rectangles have been constructed between $x = n$ and $x = 2n$, each with width 1 unit.



Let $S_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+3}} + \ldots + \frac{1}{\sqrt{2n}}$.

(α) By considering the sums of the areas of upper and lower rectangles, show that: **3**

$$2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right).$$

(β) Hence, find the value of, correct to four decimal places, **1**

$$\frac{1}{\sqrt{10^8+1}} + \frac{1}{\sqrt{10^8+2}} + \frac{1}{\sqrt{10^8+3}} + \ldots + \frac{1}{\sqrt{2\times 10^8}}.$$

End of paper.