NExT EDUCATION

2018

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10

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1 Which of the following is equivalent to $\frac{7-4i}{1-2i}$?

(A) $3+2i$

(B) $3-2i$

(C) $-7+4i$

(D) $-7-4i$

2 Find the value of the eccentricity (e) of the following equation: $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

(A) $e = \frac{6}{5}$

(B) $e = \frac{\sqrt{34}}{5}$

(C) $e = \frac{5}{3}$

(D) $e = \frac{4}{3}$

3 What are the five roots of the equation $z^5 + 1 = 0$?

(A) $z = -1,\ \text{cis}\frac{\pi}{5},\ \text{cis}\frac{3\pi}{5},\ -\text{cis}\frac{\pi}{5},\ -\text{cis}\frac{3\pi}{5}$

(B) $z = -1,\ \text{cis}\frac{\pi}{5},\ \text{cis}\frac{3\pi}{5},\ \text{cis}\left(-\frac{\pi}{5}\right),\ \text{cis}\left(-\frac{3\pi}{5}\right)$

(C) $z = -1,\ \text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ -\text{cis}\frac{2\pi}{5},\ -\text{cis}\frac{4\pi}{5}$

(D) $z = -1,\ \text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ \text{cis}\left(-\frac{2\pi}{5}\right),\ \text{cis}\left(-\frac{4\pi}{5}\right)$

4 The derivative $\frac{dy}{dx}$ of the curve $x^2 - 4xy + y^3 = 8$ is:

(A) $\frac{dy}{dx} = \frac{8(4y - 2x)}{4x - 3y^2}$

(B) $\frac{dy}{dx} = \frac{2y - x}{4x - 3y^2}$

(C) $\frac{dy}{dx} = \frac{2x - 4y}{4x - 3y^2}$

(D) $\frac{dy}{dx} = \frac{4y - 2x}{4x - 3y^2}$

5 Which of the following are the square roots of the complex number $9 - 40i$?

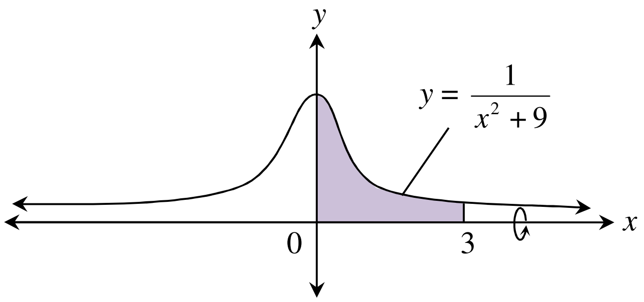
(A) $4 - 5i,\ -4 + 5i$

(B) $5 - 4i,\ -5 + 4i$

(C) $15 - 12i,\ -15 + 12i$

(D) $21 - 12i,\ -21 + 12i$

6 The shaded area enclosed by the curve $y = \dfrac{1}{x^2 + 9}$, the x-axis, the y-axis and the line $x = 3$ is shown in the diagram below.



If the shaded area is rotated about the y-axis to form a solid, which of the following would be equivalent to the volume of the solid using the method of cylindrical shells?

(A) $\pi\int_0^3 \dfrac{1}{\left(x^2+9\right)^2}\,dx$

(B) $\pi\int_0^{\frac{1}{9}} \dfrac{1}{y} - 9\,dy$

(C) $2\pi\int_0^3 \dfrac{x}{x^2+9}\,dx$

(D) $4\pi\int_0^3 \dfrac{1}{x^2+9}\,dx$

7 $P(z)$ is a polynomial of degree 4 with real coefficients.

Which of the following statements must be *false*?

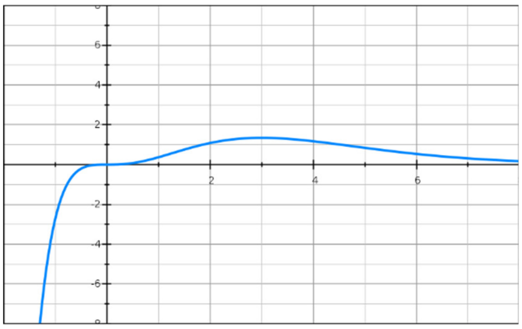
(A) $P(z)$ has four real roots.

(B) $P(z)$ has two real roots and two non-real roots.

(C) $P(z)$ has one real root and three non-real roots.

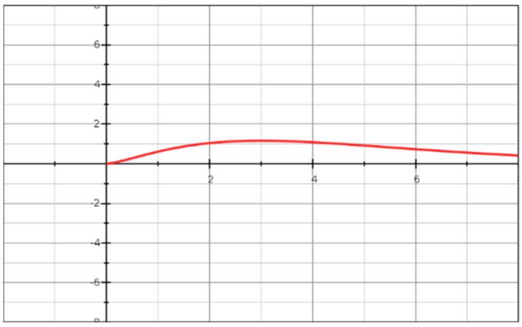
(D) $P(z)$ has no real roots.

8 The following diagram shows the graph of $y = x^3e^{-x}$:

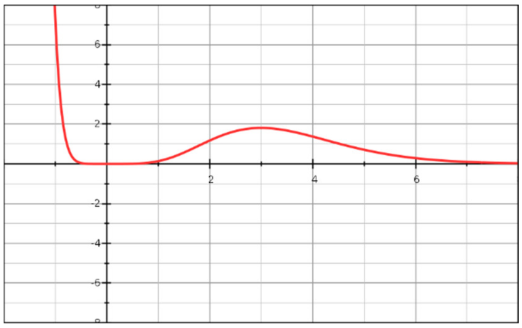


Which of the following graphs best represents $y = \log_e\left(x^3e^{-x}\right)$?

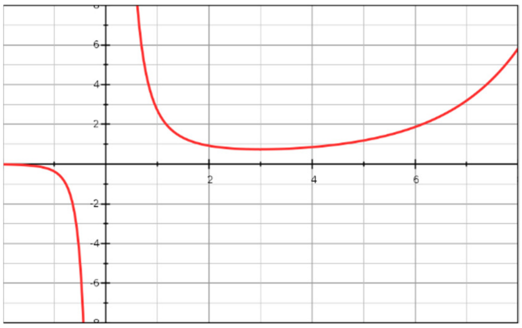
(A)



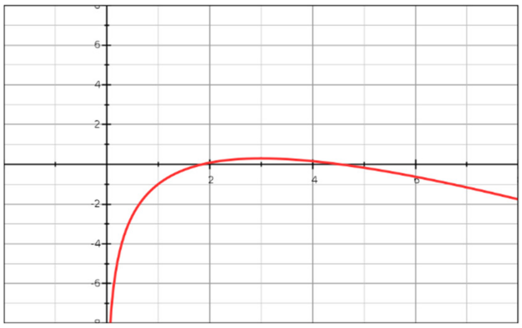
(B)



(C)



(D)



9 In how many unique ways can 12 identical balls be arranged in 5 identical boxes?

(A) $12! \times 5!$

(B) $\dfrac{16!}{12! \times 4!}$

(C) $17!$

(D) $\dfrac{17!}{12! \times 5!}$

10 If w is a complex cube root of unity, $w \neq 1$, which of the following is equivalent to the expression $\left(1 - 3w + w^2\right)\left(1 + w - 8w^2\right)$?

(A) 9

(B) 16

(C) 27

(D) 36

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question on a NEW page on your OWN PAPER.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a NEW page on your OWN PAPER.

(a) Find the real numbers x and y such that $(x+iy)(1-2i) = 8-i$. **2**

(b) (i) Find real numbers a, b and c such that **2**

$$\frac{3x^2+7x+7}{(x+3)(x^2+4)} = \frac{a}{x+3} + \frac{bx+c}{x^2+4}.$$

(ii) Hence, or otherwise, find $\int \frac{3x^2+7x+7}{(x+3)(x^2+4)}\,dx$. **2**

(c) Consider $z = 4 - 4\sqrt{3}i$.

(i) Express z in modulus-argument form. **1**

(ii) Express z^4 in the form $a + ib$, where a and b are real numbers. **2**

(d) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx$. **3**

(e) Find $\int \frac{2x-1}{\sqrt{2x-x^2}}\,dx$. **3**

End of Question 11.

Question 12 (15 marks) Use a NEW page on your OWN PAPER.

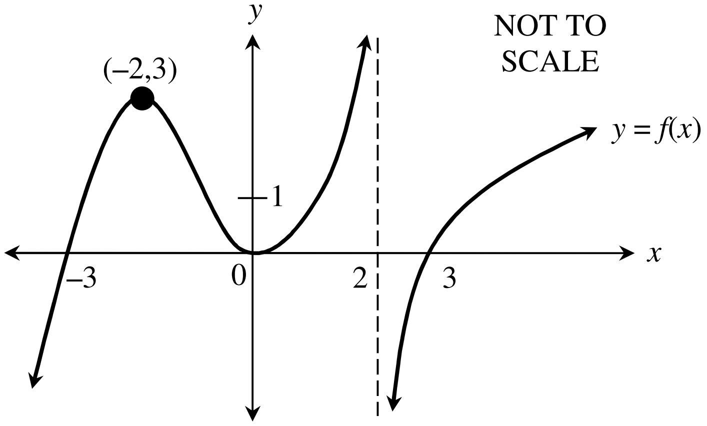
(a) For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$,

(i) Find the value of its eccentricity. **1**

(ii) Find the coordinates of its foci. **1**

(iii) Find the equations of its directricies. **1**

(b) The diagram shows the graph of a function $f(x)$, where there are stationary points at (–2,3) and (0,0), and a vertical asymptote at $x = 2$.



Sketch the following curves on separate half-page diagrams.

(i) $y = \left|f\left(x\right)\right|$ **1**

(ii) $y = \dfrac{1}{f\left(x\right)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

Question 12 continues on the next page.

(c) If α, β and γ are the roots of the equation $x^3 - x^2 + 5x + 6 = 0$, find an equation with roots of $(\alpha + \beta)$, $(\beta + \gamma)$ and $(\gamma + \alpha)$. **2**

(d) If $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a root of the equation $z^4 + 2z^3 + z^2 - 1 = 0$, find the other roots. **3**

End of Question 12.

Question 13 (15 marks) Use a NEW page on your OWN PAPER.

(a) On an Argand diagram, shade the region where $|z-1| \geq 1$ and $-\frac{\pi}{4} \leq \text{Arg}z \leq \frac{\pi}{4}$. **3**

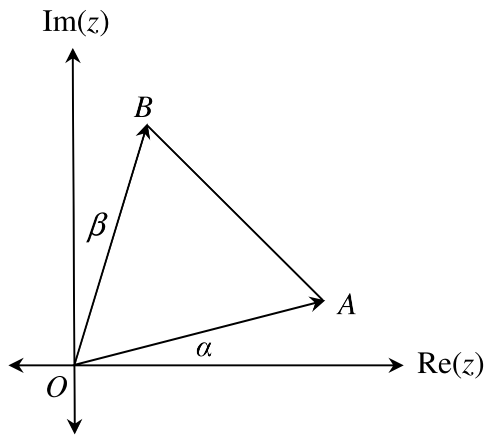
(b) Use integration by parts to find $\int e^x \sin 2x \, dx$. **3**

(c) α, β and γ are the roots of the equation $9x^3 - 36x^2 + 44x - 16 = 0$, where $\alpha = \beta + \gamma$.

(i) Find the value of α. **2**

(ii) Hence, or otherwise, fully factorise $P(x) = 9x^3 - 36x^2 + 44x - 16$. **2**

(d) On the Argand diagram, the O, A and B are the vertices of an equilateral triangle. O is the origin, and A and B are represented by the complex numbers α and β respectively, as shown in the diagram below.



Let $\theta = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Find an expression for $\overrightarrow{BA}$ in terms of α and β. **1**

(ii) Show that $\alpha = \theta(\alpha - \beta)$. **2**

(iii) Hence, or otherwise, prove that $\alpha^2 + \beta^2 = \alpha\beta$. **2**

End of Question 13.

Question 14 (15 marks) Use a NEW page on your OWN PAPER.

(a) (i) If $p > 0$ and $q > 0$, show that: $\dfrac{p}{q} + \dfrac{q}{p} \geq 2.$ **1**

(ii) If $p > 0$ and $q > 0$, show that: $(p + q)\left(\dfrac{1}{p} + \dfrac{1}{q}\right) \geq 4.$ **1**

(iii) Hence, or otherwise, if $p > 0$, $q > 0$ and $r > 0$, show that: **3**

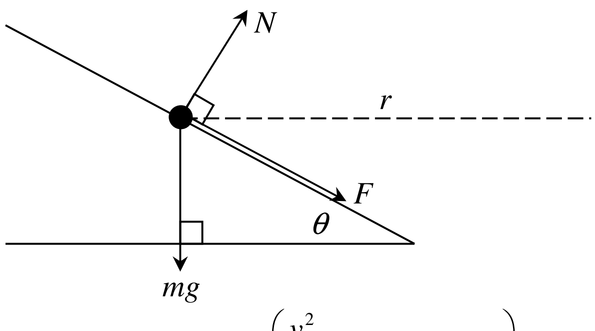
$$(p + q + r)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) \geq 9.$$

(b) (i) Let $I_n = \displaystyle\int_0^t \frac{1}{\left(1 + x^2\right)^n}\,dx$ for integers $n \geq 1$. **4**

Show that $2nI_{n+1} = (2n - 1)I_n + \dfrac{t}{\left(1 + t^2\right)^n}$ for integers $n \geq 1$.

(ii) Hence, find the value of I_3 in terms of t. **2**

(c) A vehicle of mass m kilograms moving with velocity v metres per second is rounding a bend with radius r metres banked at an angle of θ. A lateral (sideways) force F is acting between its tyres and the road, and a normal reaction force N is acting on the tyres, as shown in the diagram.



(i) By resolving forces, show that $F = m\left(\dfrac{v^2}{r}\cos\theta - g\sin\theta\right).$ **3**

(ii) Find an expression for v in terms of r, g and θ if the lateral force F is zero. **1**

End of Question 14.

Question 15 (15 marks) Use a NEW page on your OWN PAPER.

(a) An object P of mass m kilograms falls vertically from rest from a point A. The particle travels through a medium with resistance of mkv, where k is a positive constant and v is the particle's velocity in metres per second after t seconds. Assume gravity is g m/s^2.

(i) Show that $v = \frac{g}{k}\left(1 - e^{-kt}\right)$. **3**

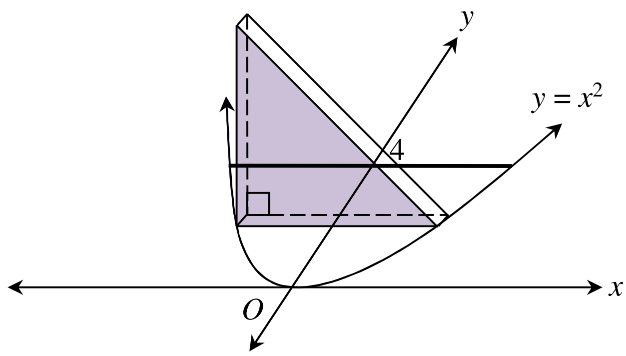
The terminal velocity of object P is V m/s.

At the same time as P begins to fall, another object Q is projected vertically upwards with initial velocity U m/s and experiences the same resistance as P.

(ii) When object Q comes to a rest, show that the velocity of P is $\frac{VU}{V+U}$. **3**

(b) A solid shape is formed on top of the curve $y = x^2$ between $y = 0$ and $y = 4$. **3**

The vertical cross-sections are right-angled isosceles triangles, where the base of each triangle is perpendicular to the y-axis and is equal to the height of the triangle. This is shown in the diagram below.



Find the volume of the solid.

Question 15 continues on the next page.

(c) (i) Using De Moivre's theorem, or otherwise, show that: **2**

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

(ii) Hence, find all the roots of $x^4 - 10x^2 + 5 = 0$ leaving your solutions in the form $x = \tan\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. **2**

(iii) Hence, find the exact value of $\tan\frac{\pi}{5}$. **2**

End of Question 15.

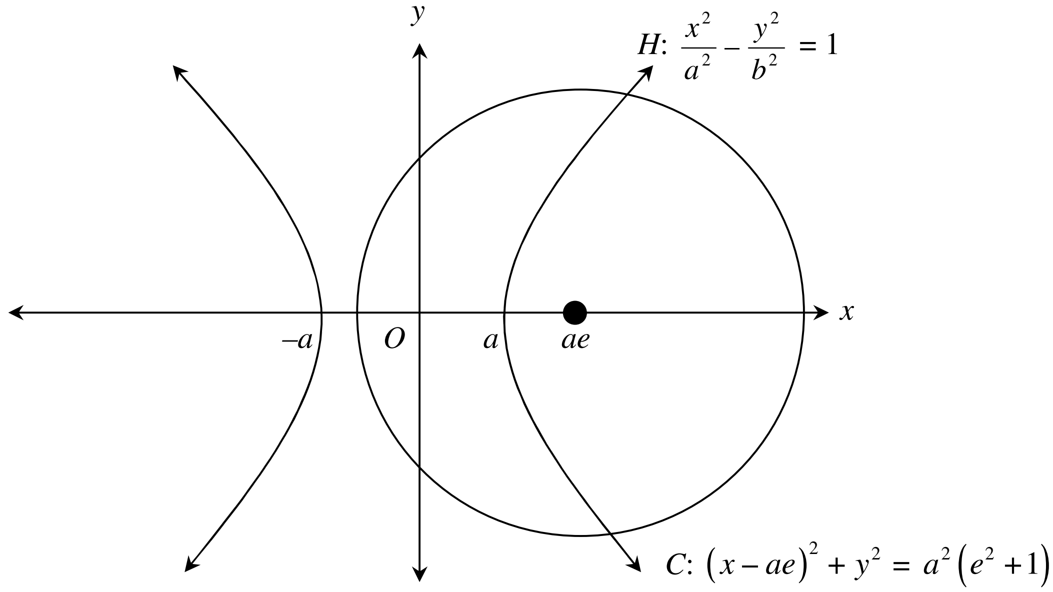
Question 16 (15 marks) Use a NEW page on your OWN PAPER.

(a) A three-digit code is made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 where no digit is repeated.

(i) Find the number of different codes that can be formed. **1**

(ii) How many of these code numbers can be formed such that the three digits do not occur in increasing order of magnitude when reading from left to right. **3**

(b) The diagram below shows the graph of a hyperbola H with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and a circle C with equation $(x - ae)^2 + y^2 = a^2(e^2 + 1)$, where e is the eccentricity of H.



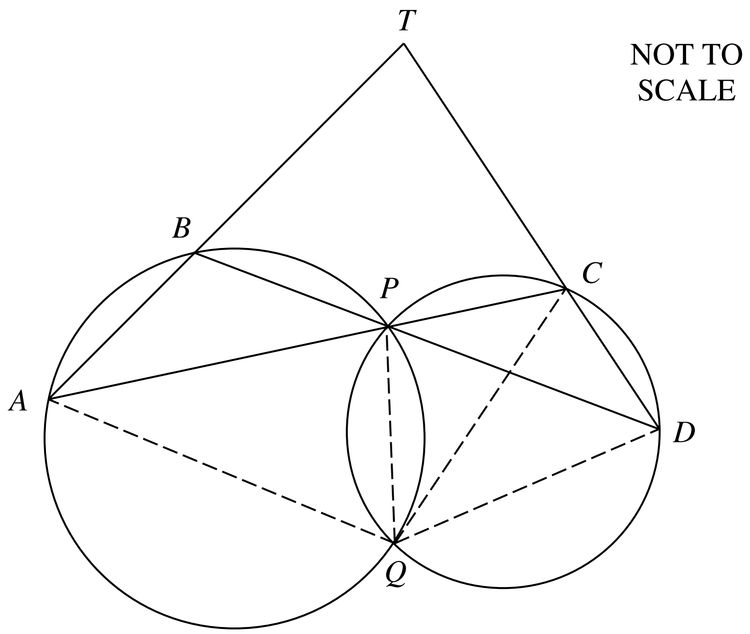
The point $P(a\sec\theta, b\tan\theta)$ lies on H, where the tangent at P has the equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. (**DO NOT PROVE THIS**). The tangent at P is also a tangent to the circle C, i.e. it is a common tangent.

(i) Show that $\sec\theta = -e$. **4**

(ii) Q is another point that lies on H and has the same properties of P, i.e. it is also a common tangent to H and C. Show that the coordinates P and Q are the extremities of a latus rectum (chord that is parallel to the directrix and passes through the focus) of H. **2**

Question 16 continues on the next page.

(c) In the diagram, two circles are drawn intersecting at P and Q. From the points A and B on the arc of the larger circle, lines are drawn through P to meet the circumference of the smaller circle at C and D respectively. The lines AB and DC produced meet at T.



(i) Let $\angle BAQ = \alpha$. **2**

Prove that $AQCT$ is a cyclic quadrilateral.

(ii) Let $\angle AQP = \beta$. **3**

If $TBPC$ is a cyclic quadrilateral, show that the points A, Q and D are collinear.

End of paper.