NEXT EDUCATION

2020

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- All answers should be written on this examination paper.
- There is some extra writing space at the end of this paper.

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 24

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Circle the best solution below for Questions 1 – 10

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1 Which of the following is $\log_5 7$ rounded to three significant figures?

(A) 0.83

(B) 0.827

(C) 1.21

(D) 1.209

2 Which of the following are the solutions for x: $x^4 - 16 = 0$?

(A) $x = -4, 4$

(B) $x = -4, -2, 2, 4$

(C) $x = -2, 2$

(D) $x = 2$ only

3 If $\dfrac{1}{3-\sqrt{5}} = p + q\sqrt{5}$, then:

(A) $p = 3, q = 5$

(B) $p = 0.75, q = 0.25$

(C) $p = -1, q = 3$

(D) $p = -3, q = 0.5$

4 Which of the following is equivalent to $\dfrac{\cos\left(\dfrac{\pi}{2}-\theta\right)}{\cos(\pi-\theta)}$?

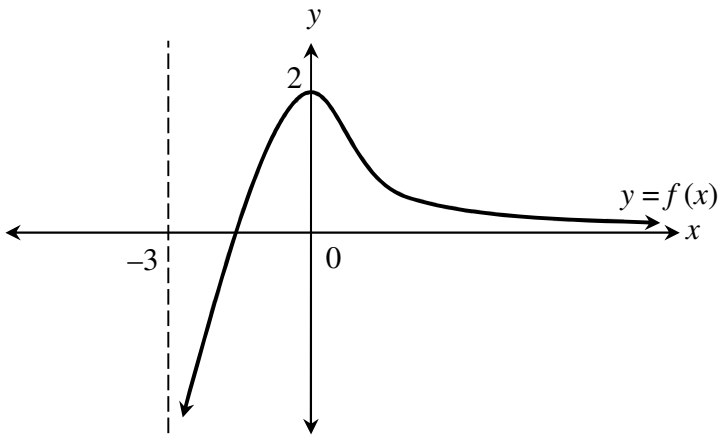
(A) $\sin\theta$

(B) $-\cos\theta$

(C) $\cot\theta$

(D) $-\tan\theta$

5 The following diagram shows the graph $y = f(x)$:



What is the range of $f(x)$?

(A) $y \in (-\infty, \infty)$

(B) $y \in (-\infty, 2)$

(C) $y \in (-\infty, 2]$

(D) $y \in [-\infty, 2]$

6 Consider the two functions $f(x) = \sqrt{x}$ and $g(x) = 2\sqrt{3 - x}$.

Which of the following best describes the types of transformation applied to the function $f(x)$ to obtain the function $g(x)$?

(A) Vertical dilation and reflection about the *y*-axis.

(B) Vertical dilation, horizontal translation and reflection about the *y*-axis.

(C) Horizontal dilation and reflection about the *x*-axis.

(D) Horizontal dilation, vertical translation and reflection about the *x*-axis.

7 The probability distribution of a discrete random variable X is summarised in the following table:

x	–1	0	1	2
$P(X = x)$	0.2	0.1	0.5	0.2

What is the value of $\text{Var}(2X - 3)$?

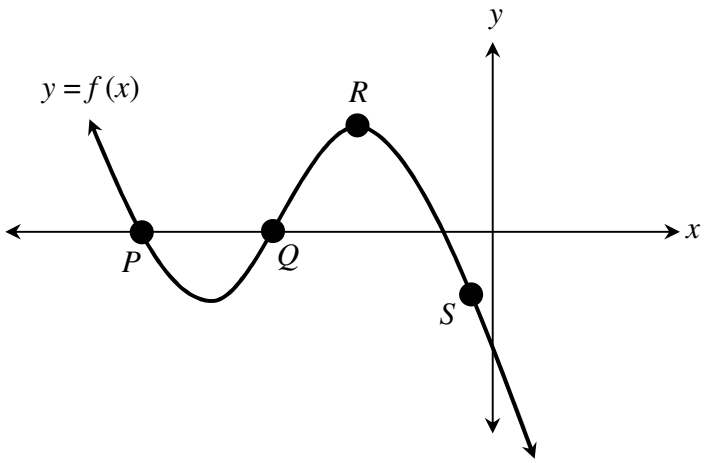
(A) 0.49

(B) 1.01

(C) 2.02

(D) 4.04

8 The following diagram shows the graph $y = f(x)$, where the coordinates P, Q, R and S lie on the curve:



At which coordinate could $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both be negative?

(A) P

(B) Q

(C) R

(D) S

9 Which of the following is the derivative of x^x?

(A) $x \times x^{x-1}$

(B) $\ln x + 1$

(C) $x^x \left(\ln x + 1\right)$

(D) $\frac{x}{\ln x}$

10 The continuous random variable X has a probability density function given by:

$$f(x) = \begin{cases} 1 - \frac{x}{2}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

The 80th percentile of X is closest to:

(A) $\frac{2}{3}$

(B) 0.586

(C) 0.778

(D) 1.106

End of Section I.

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

You should include all your solutions and working in the spaces provided in this paper.

Question 11 (15 marks)

(a) Factorise: $8x^2 - 14x - 15$. **2**

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(b) Solve for x: $\dfrac{3-2x}{x+5} \leq 0$, expressing your solution in set notation. **2**

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(c) Find the limiting sum of the series: $\dfrac{5}{6} - \dfrac{1}{2} + \dfrac{3}{10} - \dfrac{9}{50} + \ldots$ **2**

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Question 11 continues on the next page.

(d) Differentiate the following with respect to x:

(i) $y = 3x^{-5}$. **1**

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(ii) $y = (1 + \sin x)^4$. **1**

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(iii) $y = \log_e (e^{2x} + 5)$. **2**

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(e) Find the equation of the tangent to the curve $y = (x^2 + 1)^3$ at the point where $x = 1$. **3**

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(f) If $\cos\theta = \frac{3}{8}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the exact value of $\cot\theta$. **2**

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End of Question 11.

Question 12 (15 marks)

(a) Find:

(i) $\int \sqrt{x\sqrt{x}}\, dx$. **2**

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(ii) $\int \sin 2x \left(e^{\cos 2x}\right) dx$. **2**

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(iii) $\int_0^1 \frac{5x^3}{x^4 + 1}\, dx$. **2**

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Question 12 continues on the next page.

(b) The probability distribution of a discrete random variable X is summarised in the following table:

x	3	4	5	6	7
$\mathbf{P}(X = x)$	$2p$	$3p + q$	0.15	$4p - q$	0.4

where p and q are constants.

(i) Find the value of p and q given that $P(X \leq 5) = 0.5$. **2**

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(ii) Hence, find the value of $\mathrm{E}(X)$. **1**

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(c) A radioactive substance with initial mass of 3g decays over time t (in days) according to the formula:

$$M = 3e^{-kt}$$

where M is the mass at time t and k is a positive constant.

(i) Show that M satisfies the differential equation $\frac{dM}{dt} = -kM$. **1**

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Question 12 continues on the next page.

(ii) If the radioactive substance's 'half-life' (the time it takes for the substance to lose half its mass) was 7 days, find the value of the constant k, rounding your solution to three significant figures. **2**

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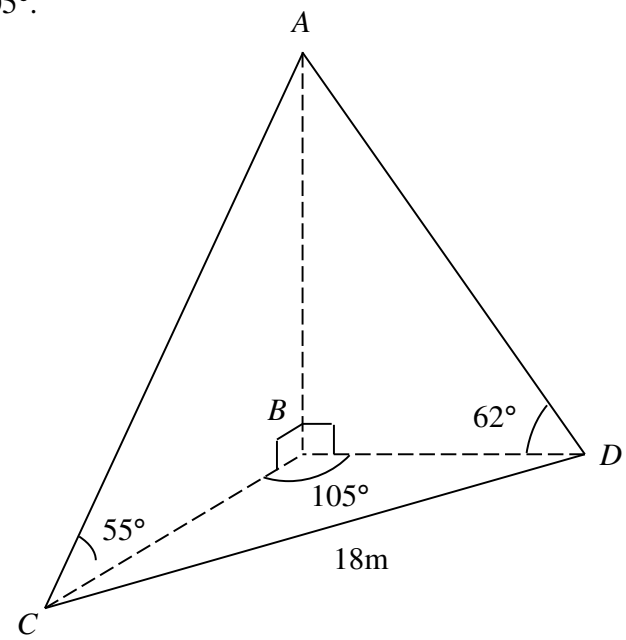
(d) The diagram shown is a triangular pyramid where $\angle ACB = 55°$, $\angle ADB = 62°$, and $\angle CBD = 105°$. **3**



If the length of CD is 18m, find the length of AB to nearest metre.

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End of Question 12.

Question 13 (15 marks)

(a) (i) Solve for x, where $0 \le x \le 2\pi$: $4\sin(2x + \pi) + 2 = 0$. **3**
Leave your solution in exact form.

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(ii) Draw a neat sketch of $y = 4\sin(2x + \pi) + 2$, where $0 \le x \le 2\pi$, showing all intercepts. **2**

(b) Consider the function: $f(x) = -3x^2 + 12x - 7$

(i) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. **2**

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Question 13 continues on the next page.

(ii) Describe the transformations made to the function $y = x^2$ to result in $f(x)$. **3**

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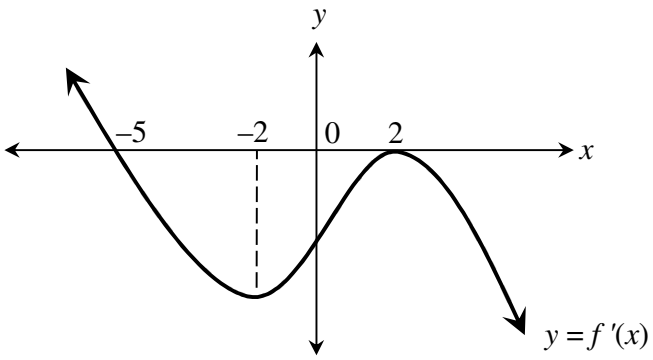
(c) The following diagram shows the graph $y = f'(x)$:



(i) State the values of x where the graph $y = f(x)$ is increasing? **1**

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(ii) Draw a neat sketch of $y = f''(x)$, showing all key features. **2**

Question 13 continues on the next page.

(iii) Given that the y-intercept of the graph $y = f(x)$ is 7, draw a neat sketch of $y = f(x)$, showing all key features. **2**

End of Question 13.

Question 14 (15 marks)

(a) Consider the functions $f(x) = \sqrt{x}$ and $g(x) = 2 - x$.

(i) Find the value of $f\left[g\left(-7\right)\right]$. **1**

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(ii) Solve for x: $f\left(x\right) = g\left(x\right)$. **2**

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(iii) Find the domain and range of $y = g\left[f\left(x\right)\right]$ in set notation. **2**

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(b) The displacement x cm of a of a particle moving along a straight line after time t seconds is given by the formula:

$$x = 3t + e^{-3t}.$$

(i) Find the position of the particle when $t = 1$, rounding to two decimal places. **1**

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(ii) By finding an expression for the velocity of the particle in terms of t, show that the particle is initially at rest. **2**

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Question 14 continues on the next page.

(iii) Find an expression for the acceleration of the particle in terms of t. **1**

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(iv) Find the limiting velocity of the particle as $t \to \infty$. **1**

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(c) Simplistically, an annuity product comprises of a series of payments paid to an investor periodically (e.g. monthly or annually) in return for an initial upfront investment amount.

(i) An annuity with an initial price of \$$P$ paid \$$M$ to the investor at the end of every year for n years. If the investment rate of return was r % p.a., show that the initial price of the annuity is given by the formula: **2**

$$P = \frac{M\left[1 - \frac{1}{(1+r)^n}\right]}{r}$$

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Question 14 continues on the next page.

(ii) Hence, or otherwise, find the initial price of an annuity that paid \$25000 at the end of every year for a fixed term of 15 years, assuming that the investment rate of return was 7.5% p.a. Round your solution to the nearest dollar. **1**

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A university wants to set up a scholarship that worked like an annuity, providing a fixed payment at the end of each year to a student selected based on their performance during the year (aka. the 'scholar') . The university is keen to have this scholarship in place continuing for a long time, providing the fixed payment to future scholars indefinitely. To do this, it utilises a 'perpetuity' concept that basically involves setting up a one-off deposit into an investment fund, where they can withdraw the fixed amount at the end of each year indefinitely.

(iii) Using the formula in (i), show that the initial investment amount of \$$P$ for a perpetuity paying \$$M$ periodically indefinitely at an investment rate of return of r % p.a. is given by the formula: **1**

$$P = \frac{M}{r}$$

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(iv) Hence, or otherwise, find the amount that the university needs to deposit into the fund with an investment rate of return of 3.6% p.a. to fund a scholarship program paying \$18,000 annually into the foreseeable future. Round your solution to the nearest dollar. **1**

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End of Question 14.

Question 15 (15 marks)

(a) The probability density function of a continuous random variable X is given by:

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Find the value of k. **1**

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Using part (i),

(ii) Find the mean and variance. **2**

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(iii) Find the median, rounding your answer to three decimal places. **1**

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(b) Find the value of p and q: $\log_3 2 + \log_9 16 = \log_p q$. **3**

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Question 15 continues on the next page.

(c) Consider the curve $y = \dfrac{x+1}{(x-1)^2}$.

(i) Find the domain of the curve, stating your solution in set notation. **1**

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(ii) Find any coordinates of stationary points and, if they exist, determine their nature. **3**

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Question 15 continues on the next page.

(iii) Given that the second derivative to the curve is $\frac{d^2y}{dx^2} = \frac{2x+10}{(x-1)^4}$. **1**

(**DO NOT PROVE THIS RESULT**).

Show that there is a point of inflexion at $x = -5$.

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(iv) Explain the behaviour of the curve for large values of x. **1**

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(v) Sketch the curve $y = \frac{x+1}{(x-1)^2}$. **2**

End of Question 15.

Question 16 (15 marks)

(a) Evaluate $\int 3\left(5^x\right) dx$. **2**

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(b) The value of diamonds is commonly known to increase relative to its size measured in 'carats' (equal to 200mg). A jeweller received a new shipment of twelve diamonds of varying sizes. The following table is a summary of the value the jeweller placed on each of the twelve diamonds from this shipment.

Diamond no.	Size of diamond (in carats)	Value of diamond (in $'000)
1	0.6	4.1
2	1.3	24.8
3	2.5	55.4
4	1.7	30.3
5	1.6	28.9
6	0.5	3.6
7	1.0	17.2
8	1.2	22.1
9	2.2	40.2
10	3.5	70.5
11	2.8	66.3
12	1.5	26.5

Question 16 continues on the next page.

By using the table above,

(i) Determine the mean and standard deviation of the size of diamonds in this shipment, rounding your solutions to three decimal places. **2**

Mean = ☐ Standard deviation = ☐

By applying the method of least squares regression to the data in the table above,

(ii) State the independent variable. **1**

Independent variable = ☐

(iii) Determine the value of Pearson's correlation coefficient (r), rounding your solution to three decimal places. **1**

r = ☐

(iv) Write in the missing values that represent the equation of the 'line of best fit' in the spaces provided. Round your solution to three decimal places. **2**

Value of diamond = ☐ + ☐ × Size of diamond

where the value of diamond is in $,000 and the size of diamond is in carats.

(v) Describe any practical limitations of applying the equation in (v) to valuing relatively small diamonds. **1**

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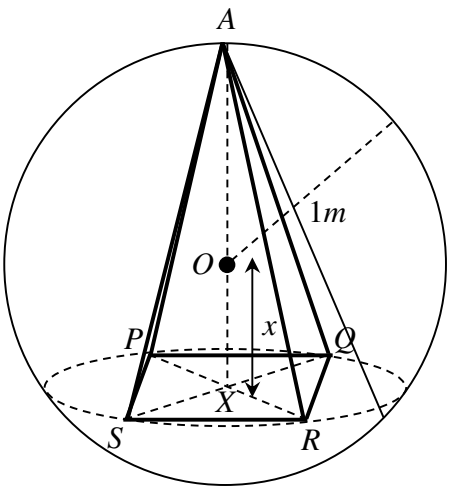
Question 16 continues on the next page.

(c)



In the diagram above, a square pyramid with base of $PQRS$ and apex A is inscribed within a sphere with radius of 1 metre, where all vertices of the pyramid lie on the sphere's surface. The diagonals PR and QS meet at X and $OX = x$ metres.

(i) Show that the volume, V, of the pyramid is given by $V = \frac{2}{3}\left(1 + x - x^2 - x^3\right)$. **3**

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Question 16 continues on the next page.

(ii) Hence, or otherwise, find the maximum volume of the pyramid, rounding your answer to three decimal places. **3**

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