Student details

Name: ______________________

Mark: ______________________

2022

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Reference sheet is provided separately.
- Marks may be lost for poor working out and/or poor logic.

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Circle the BEST solution.

Section II Pages 6 – 13

90 marks

- Attempt Questions 11 – 33
- Your responses should include relevant mathematical reasoning and/or calculations.

Section I

10 marks
Attempt Questions 1 – 10

<u>Circle the BEST solution</u> below for Questions 1 – 10.

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1 For the statement "*if roses are red then violets are blue*", which of the following represents the statement's contrapositive?

(A) "*If violets are not blue then roses are red.*"

(B) "*If roses are red then violets are not blue.*"

(C) "*If violets are not blue then roses are not red.*"

(D) "*If roses are not red then violets are not blue.*"

2 Which of the following is the unit vector of $2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$?

(A) $3\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

(B) $\frac{1}{\sqrt{3}}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(C) $\sqrt{3}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(D) $\frac{1}{3}\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

3 If $w = \sqrt{3} + i$, which of the following is equivalent to w^8 in exponential form?

(A) $e^{-\frac{\pi}{4}i}$

(B) $8e^{\frac{5\pi}{6}i}$

(C) $128e^{\frac{3\pi}{4}i}$

(D) $256e^{-\frac{2\pi}{3}i}$

4 Which of the following is the equation of the vector passing through the point (4,–1) and perpendicular to the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, where λ is a constant?

(A) $\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(B) $\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

(C) $\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(D) $\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

5 If $p, q \in \mathbb{R}^+$ and $p > q$, which one of the following statements might be false?

(A) $\dfrac{1}{p - q} > 0$

(B) $\dfrac{p}{q} - \dfrac{q}{p} > 0$

(C) $p + q > 2q$

(D) $2p > 3q$

6 What is the equivalent to $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$?

(A) $-4a\sqrt{x^2 - a^2} + c$

(B) $\cos^{-1}\left(\frac{x}{a}\right) + c$

(C) $\sin^{-1}\left(\frac{x}{a}\right) + c$

(D) $\ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c$

7 Consider the region $|z - 2| \leq 1$ on the Argand diagram. What is the maximum value of $|z|$?

(A) 1

(B) $\sqrt{3}$

(C) 2

(D) 3

8 A particle moves with simple harmonic motion along a straight line, where its velocity v m/s is given by the formula: $v^2 = 6 + 4x - 2x^2$, where x is the particle's displacement from a fixed point O (in metres).

What is the particle's amplitude of motion?

(A) 6m

(B) 2m

(C) $\sqrt{3}$ m

(D) 12m

9 w is a complex cube root of unity and $w \neq 1$.

What is the value of $(1-w)(1-w^2)(1-w^4)(1-w^8)$?

(A) $1 - w$

(B) $1 + w^2$

(C) 8

(D) 9

10 An object of m kg is projected into the air with initial velocity of u m/s at an angle of θ against the horizontal. The object experiences gravity of g m/s^2 and air resistance proportional to its velocity (i.e. $R = mkv$, where k is a constant).

Which of the following represents the equation of the object's vertical displacement after t seconds?

(A) $y = \dfrac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$.

(B) $y = \dfrac{1}{k}\log_e\left|ku\cos t + 1\right|$.

(C) $y = \dfrac{g + ku\sin\theta}{k^2}\left(1 - e^{-kt}\right) - \dfrac{gt}{k}$.

(D) $y = \dfrac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \dfrac{ku\sin\theta}{\sqrt{gk}}\sin\left(\sqrt{gk}t\right)\right|$.

Section II

90 marks
Attempt Questions 11–33

In Questions 11–33, your responses should include relevant mathematical reasoning and/or calculations.

Question 11

Evaluate $|2-3i|$. **1**

Question 12

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$.

(a) Find $|\underset{\sim}{a}|$. **1**

(b) Find the size of the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (nearest degree). **1**

(c) Find the vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. **1**

Question 13

If $w = 3 - 2i$, express each of the following in the form $a + ib$, where a and b are real.

(a) w^{-1}. **1**

(b) $4w + \overline{w}$. **2**

Question 14

Find $\int \frac{1}{\sqrt{1-x-x^2}}\, dx$. **2**

Question 15

Find the value of the real numbers a and b such that $(a + bi)^2 = 9 + 40i$. **2**

Question 16

Use integration by parts to evaluate $\int_1^e x^4 \log_e x \, dx$. **3**

Question 17

Prove by contradiction that $\log_5 4$ is irrational. **2**

Question 18

(a) If $a, b \in \mathbb{R}$, show that: $a^2 + b^2 \geq 2ab$. **1**

(b) If $a, b, c \in \mathbb{R}$, show that: $a^2 + b^2 + c^2 \geq ab + ac + bc$. **2**

(c) Hence, or otherwise, show that: $3(a^4 + b^4 + c^4) \geq \left(a^2 + b^2 + c^2\right)^2$. **2**

Question 19

On an Argand diagram, shade the region where: **3**

$$1 \leq \operatorname{Re}(z) \leq 3 \quad \text{and} \quad -\frac{\pi}{3} \leq \operatorname{Arg}(z) \leq \frac{\pi}{6}.$$

Question 20

Prove that $\forall a \in (0, \infty), \forall b \in (0, \infty), \log_{\frac{1}{a}} \frac{1}{b} = \log_a b$. **2**

Question 21

(a) Find the value of p and q in terms of a: $\dfrac{3x^2 - ax}{(x-2a)(x^2+a^2)} = \dfrac{p}{x-2a} + \dfrac{x+q}{x^2+a^2}$ **2**

(b) Hence, or otherwise, evaluate $\displaystyle\int_0^a \frac{3x^2 - ax}{(x-2a)(x^2+a^2)}\,dx$. **2**

Question 22

If a point of intersection exists between $\underset{\sim}{r}_1 = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 6 \end{pmatrix}$ and $\underset{\sim}{r}_2 = \begin{pmatrix} 3 \\ 12 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$,

(a) Find the value of λ and μ. **2**

(b) Hence, or otherwise, find the point of intersection. **1**

Question 23

A particle is moving along a straight line where its displacement x metres from O after t seconds is given by the formula:

$$x = 5 + \sin^2 t.$$

(a) Show that the particle moves with simple harmonic motion. **2**

(b) Find the period of motion. **1**

(c) Find the total distance travelled by the particle in the first π seconds. **2**

Question 24

Consider the sphere S: $(x-1)^2+(y-2)^2+(z+1)^2=9$ with centre $C(1,2,-1)$.

(a) Show that the point $A(3,3,1)$ lies on the sphere S. **1**

(b) If A is represented by the position vector $\underset{\sim}{a}$ show that the vector $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$, **2**

where λ is a constant, is a tangent to the sphere S if $\underset{\sim}{d}=\begin{pmatrix}4\\0\\-4\end{pmatrix}$.

Question 25

(a) Show that: $2\cos A\sin B=\sin(A+B)-\sin(A-B)$. **1**

(b) Show that: $\dfrac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}=\cos x$. **2**

(c) Hence, or otherwise, prove by mathematical induction for $n\in\mathbb{Z}^+$: **4**

$$\cos x+\cos 2x+\cos 3x+\ldots+\cos nx=\frac{\sin\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}-\frac{1}{2}.$$

Question 26

Let $z=e^{i\theta}$.

(a) Show that $z^n-\dfrac{1}{z^n}=2i\sin(n\theta)$. **1**

(b) Show that $\left(z-\dfrac{1}{z}\right)^5=\left(z^5-\dfrac{1}{z^5}\right)-5\left(z^3-\dfrac{1}{z^3}\right)+10\left(z-\dfrac{1}{z}\right)$. **1**

(c) Hence, find $\displaystyle\int\sin^5\theta\,d\theta$. **3**

Question 27

(a) Using De Moivre's theorem, show that: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **1**

(b) Hence, or otherwise, find all the roots of $8x^3 - 6x - 1 = 0$ leaving your solutions in the form $x = \cos\theta$. **2**

(c) Hence, show that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$. **2**

Question 28

An object of mass m kg is projected vertically upwards from the ground through gravity of g m/s^2. It experiences air resistance of $\frac{1}{100}mgv^2$, where v m/s is the velocity of the object.

The object's initial velocity is u m/s and its displacement after t seconds is x metres.

(a) Show that the upward motion of the object is $a = -\frac{g}{100}\left(100 + v^2\right)$. **1**

(b) Show that the object's greatest height above the ground is $\frac{50}{g}\log_e\left(\frac{100 + u^2}{100}\right)$ m. **2**

Once the object reaches its maximum height, it falls back towards the ground. Let x now be the downward displacement from where the object reaches maximum height.

(c) Show that the downward motion of the object is $a = \frac{g}{100}\left(100 - v^2\right)$. **1**

(d) Find the terminal velocity V of the object for the downward motion. **1**

(e) Show that the velocity when the object reaches the ground is $\frac{10u}{\sqrt{100 + u^2}}$ m/s. **2**

(f) If the object's initial was V m/s, show that its velocity when it returns to the ground Is $\frac{1}{\sqrt{2}}V$ m/s. **1**

Question 29

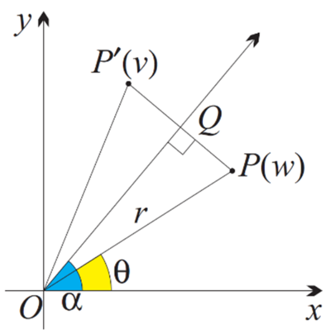
Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for integers $n \geq 2$.

(a) Show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for integers $n \geq 2$. **3**

(b) Hence, find the value of $\displaystyle\int_0^2 \left(4 - y^2\right)^{\frac{5}{2}} dy$. **3**

Question 30

In the diagram below, the point P represents a complex number $w = r \operatorname{cis} \theta$, while the point Q is the point on the ray $\arg(z) = \alpha$ such that $\angle PQO = \dfrac{\pi}{2}$. The point P', which represents the complex number v, is a reflection of P about the ray $\arg(z) = \alpha$.



You may assume that $\Delta OPQ \equiv \Delta OP'Q$.

(a) Write down the values of $|v|$ and $\arg(v)$. **2**

(b) Hence, show that $v = \overline{w} \operatorname{cis} 2\alpha$ **1**

(c) The circle $|z - (2 + 2i)| = 1$ is reflected in the ray $\arg(z) = \dfrac{\pi}{6}$. By using the result (b), or otherwise, show that the equation of this new circle is **2**

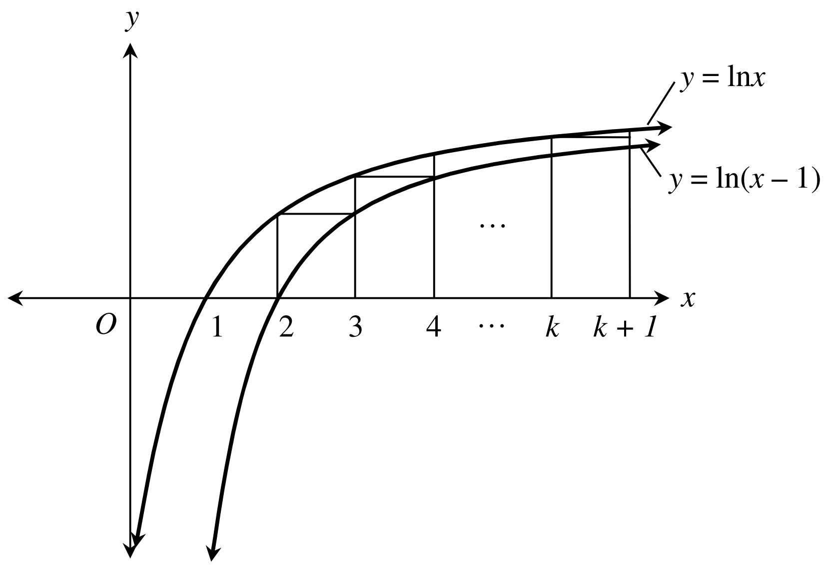
$$\left| z - \left[\left(1 + \sqrt{3}\right) + i\left(\sqrt{3} - 1\right) \right] \right| = 1$$

Question 31

Using the substitution $u = \dfrac{1}{x}$, or otherwise, evaluate $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2+2x-1}}\,dx$. **3**

Question 32

In the diagram below, $k - 1$ rectangles are constructed from $x = 2$ to $x = k + 1$, where $k \geq 2$, between the graphs of $y = \ln x$ and $y = \ln(x-1)$. **4**



Show that: $k^k < k!e^{k-1} < \dfrac{1}{4}(k+1)^{k+1}$ where $k \geq 2$.

Question 33

Consider two numbers a_1 and a_2, where $a_1, a_2 \in \mathbb{R}^+$.

(a) Prove that: $\dfrac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$. **1**

Let $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$.

If $a_1 a_2 \ldots a_n = 1$ then $a_1 + a_2 + \ldots + a_n \geq n$. (Do NOT prove this.)

(b) Prove that: $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$. **2**

(c) Hence, or otherwise, prove that for integers $n \in [1, \infty)$: $2^n - 1 > n\sqrt{2^{n-1}}$ **3**

End of paper.