Student details

Name: ____________________

Mark: ____________________

2021

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Reference sheet is provided separately.
- Marks may be lost for poor working out and/or poor logic.

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Circle the BEST solution.

Section II Pages 6 – 13

90 marks

- Attempt Questions 11 – 33
- Your responses should include relevant mathematical reasoning and/or calculations.

Section I

10 marks
Attempt Questions 1 – 10

Circle the BEST solution below for Questions 1 – 10.

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1 What are the solutions for z in the equation $z^2 + z + 6 = 0$?

(A) $z = 3, -2$

(B) $z = 2i, 3i$

(C) $z = \dfrac{-1 \pm \sqrt{35}}{2}$

(D) $z = \dfrac{-1 \pm i\sqrt{23}}{2}$

2 What is the negation of the statement $\exists\, n \in \mathbb{R},\ n \le k$?

(A) $\exists\, n \notin \mathbb{R},\ n \le k$

(B) $\exists\, n \in \mathbb{R},\ n > k$

(C) $\forall\, n \in \mathbb{R},\ n \le k$

(D) $\forall\, n \in \mathbb{R},\ n > k$

3 Which of the following equations DOES NOT represent simple harmonic motion, where x is displacement after t seconds?

(A) $x = 3\cos t$

(B) $x = 9\sin 4t$

(C) $x = 5\tan t$

(D) $x = 3\sin 2t + 4\cos 2t$

4 If $z = 1 - i\sqrt{3}$, which of the following is equivalent to z^4 in exponential form?

(A) $e^{\frac{2\pi}{3}i}$

(B) $e^{-\frac{2\pi}{3}i}$

(C) $16e^{\frac{2\pi}{3}i}$

(D) $16e^{-\frac{2\pi}{3}i}$

5 What is the size of the acute angle between the vectors $\underset{\sim}{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$?

(A) 17°

(B) 21°

(C) 33°

(D) 34°

6 What is the equivalent to $\int \frac{1}{x^2 + x + 1}\, dx$?

(A) $\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2x+1)}{3}\right) + c$

(B) $\frac{1}{2}\log_e\left(\frac{2x+1}{2x-1}\right) + c$

(C) $\sin^{-1}\left(\frac{2\sqrt{3}x+1}{3}\right) + c$

(D) $\left(\sqrt{2x+1}\right)^3 + c$

7 Which of the following is equivalent to $\left(\sqrt{4+3i}+\sqrt{4-3i}\right)^2$?

(A) $2\left(\sqrt{4+3i}+\sqrt{4-3i}\right)$

(B) 25

(C) 18

(D) 10

8 What is the equivalent to $\int \frac{29\sin x-22\cos x}{4\sin x+3\cos x}\,dx$?

(A) $\frac{6}{4\sin x+3\cos x}+c$

(B) $3\cos x-4\sin x+c$

(C) $2x-7\log_e\left|4\sin x+3\cos x\right|+c$

(D) $3\left(4\sin x+3\cos x\right)^2+c$

9 A particle with mass of 1kg is projected vertically upwards from the ground with a velocity of 60 metres per second through air resistance of $\frac{v^2}{100}$.

Assuming gravity of 10 metres per second^2, which of the following is an expression for the maximum height the particle can attain?

(A) $50\log_e\left(\frac{23}{5}\right)$ metres

(B) $60\tan^{-1}\left(\frac{6}{5}\right)$ metres

(C) 32 metres

(D) $30e^{\frac{25}{24}}$ metres

10 The tide at a harbour generally follows simple harmonic motion, where high tide is measured at 11 metres and low tide is measured at 5 metres. Every 26 hours, two complete periods of motion are observed.

On the first day of this month, the low tide was observed at 5am. What is the latest time tomorrow at which the tide is increasing at the fastest rate?

(A) Tomorrow at 10:15am.

(B) Tomorrow at 11:55am.

(C) Tomorrow at 9:45pm.

(D) Tomorrow at 11:15pm.

Section II

90 marks
Attempt Questions 11–32

In Questions 11–32, your responses should include relevant mathematical reasoning and/or calculations.

Question 11

Find the value of i^{2021}. **1**

Question 12

If $z = 6i - 8$, express each of the following in the form $a + ib$, where a and b are real.

(a) $z\bar{z}$. **1**

(b) z^{-1}. **1**

(c) $\sqrt{z}$. **2**

(d) $(z + 2)^6$. **2**

Question 13

Use integration by parts to find $\int e^{-x} \sin 5x \, dx$. **2**

Question 14

Find the vector projection of $\underset{\sim}{u} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$ in the direction of $\underset{\sim}{v} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$. **2**

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Question 15

Using Question 12, or otherwise, solve for z expressing your solution in the form a + ib, where a and b are real: **2**

$$2z^2 - (3 + i)z + 2 = 0.$$

Question 16

Prove the following statement by contradiction: **3**

$$\forall p \in \mathbb{R},\, p \notin \mathbb{Q} \Rightarrow (2p + 1) \notin \mathbb{Q}.$$

Question 17

(a) Find real numbers b and c such that $\dfrac{9x^2 + x + 50}{(x^2+9)(x-1)} = \dfrac{3x + c}{x^2+9} + \dfrac{b}{x-1}$. **2**

(b) Hence, or otherwise, find $\displaystyle\int \frac{9x^2 + x + 50}{(x^2+9)(x-1)}\,dx$. **3**

Question 18

Find the centre and radius of the sphere with equation $x^2 + y^2 + z^2 - 4x + z - 3 = 0$ **2**

Question 19

On an Argand diagram, shade the region where: **3**

$$|z - 2 + 2i| \geq 2, \quad -\frac{\pi}{4} \leq \operatorname{Arg} z \leq 0 \quad \text{and} \quad z + \bar{z} < 8.$$

Question 20

If $p > 0$, $q > 0$, $r > 0$ and $s > 0$,

(a) Show that: $p^2 + q^2 \geq 2pq$. **1**

(b) Show that: $p^4 + q^4 + r^4 + s^4 \geq 4pqrs$. **2**

(c) Show that: $(p^2 + q^2)^2 + (r^2 + s^2)^2 \geq 2(pq + rs)^2$. **2**

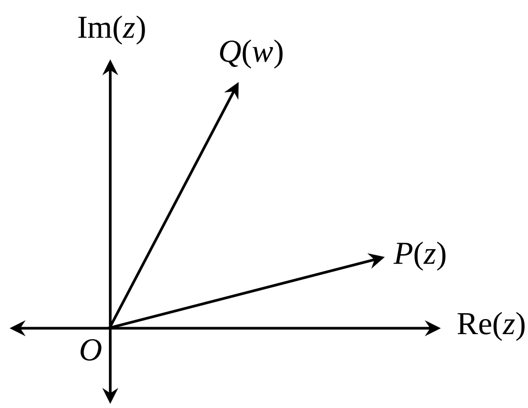
Question 21

(a) Find the vector equation of the straight line that passes through the point $(-4, 5, 9)$ and is parallel to the vector $\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$. **1**

(b) Find the point of intersection between the vectors $\underset{\sim}{u} = \left(\underset{\sim}{i} + 4\underset{\sim}{j}\right) + \lambda\left(2\underset{\sim}{i} - \underset{\sim}{j}\right)$ and $\underset{\sim}{v} = \left(3\underset{\sim}{i} + 9\underset{\sim}{j}\right) + \mu\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)$, where λ and μ are constants. **2**

Question 22

Points P and Q are represented by the complex numbers z and w on the Argand diagram, where $|z| = |w| = 2$, as shown in the diagram below. **3**



If $\text{Arg}z = \theta$ and $\text{Arg}w = \phi$, show that $|z + w| = 4\cos\left(\dfrac{\phi - \theta}{2}\right)$.

Question 23

Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$. **3**

Question 24

Two points, A and B, move on the number plane according to the equations $y = 4 + 3\cos 2t$ and $x = 3 + 2\sin t$ respectively, where t is in seconds.

(a) Show that points A and B both move with simple harmonic motion. **2**

(b) Find the rate at which the area of triangle OAB is changing at $t = \frac{5\pi}{4}$ seconds, where O is the origin. **3**

Question 25

Let $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\, dx$ for integers $n \geq 0$.

(a) Find $\int \sec x\, dx$. **1**

(b) Show that $I_n = \frac{\sqrt{2}^{\,n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ for integers $n \geq 2$. **3**

(c) Hence, find the value of $\int_0^1 \left(1 + y^2\right)^{\frac{3}{2}} dy$. **3**

Question 26

(a) Using De Moivre's theorem, show that: $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$. **1**

(b) Hence, or otherwise, find all the roots of $x^3 - 3x^2 - 3x + 1 = 0$ leaving your solution in the form $x = \tan\theta$. **2**

(c) Hence, show that $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12} = 4$. **2**

Question 27

A particle of mass m is falling through a medium with resistance mkv^2, starting from rest. Assuming gravity of g m/s^2,

(a) Show that the particle's terminal velocity V is $\sqrt{\dfrac{g}{k}}$. **1**

(b) After t seconds, the particle has travelled x metres and attained a velocity of v ms^{-1}.

(i) Show that: $x = \dfrac{1}{2k}\log_e\left(\dfrac{g}{g - kv^2}\right)$. **2**

(ii) Show that: $v = V\left(\dfrac{e^{\frac{2gt}{V}} - 1}{e^{\frac{2gt}{V}} + 1}\right)$. **3**

Question 28

Solve for θ, where $-\pi \le \theta \le \pi$: $\left|e^{2i\theta} + e^{-2i\theta}\right| = 1$. **3**

Question 29

A projectile of mass m kilograms was launched through a medium with an initial velocity of 60 m/s at an angle of 30° to the horizontal ground. In addition to gravity of 10 m/s^2, the projectile experiences air resistance proportional to the projectile's velocity v of $\frac{mv}{200}$. The horizontal and vertical equations of the cannonball's motion are given as follows:

$$\ddot{x} = -\frac{\dot{x}}{200} \qquad \ddot{y} = -10 - \frac{\dot{y}}{200}$$

$$\dot{x} = 30\sqrt{3}e^{-\frac{t}{200}} \qquad \dot{y} = 2030e^{-\frac{t}{200}} - 2000$$

$$x = 6000\sqrt{3}\left(1 - e^{-\frac{t}{200}}\right) \qquad y = 406000\left(1 - e^{-\frac{t}{200}}\right) - 2000t$$

(a) The projectile takes approximately 6 seconds to return back to the ground. (**DO NOT PROVE THIS**). **2**

Verify this result, and find the horizontal distance travelled by the projectile, rounding your answer to the nearest metre.

(b) Find the time it takes for the projectile to reach its maximum height, rounding your answer to two decimal place. **1**

(c) Using the rounded solution in part (b), find the maximum height attained by the projectile, rounding your answer to the nearest metre. **1**

Question 30

(a) Express $1 + x + x^2 + x^3 + x^4 + x^5$ as a product of real factors. **1**

(b) Hence, or otherwise, show that the equation **3**

$$k + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} = 0$$

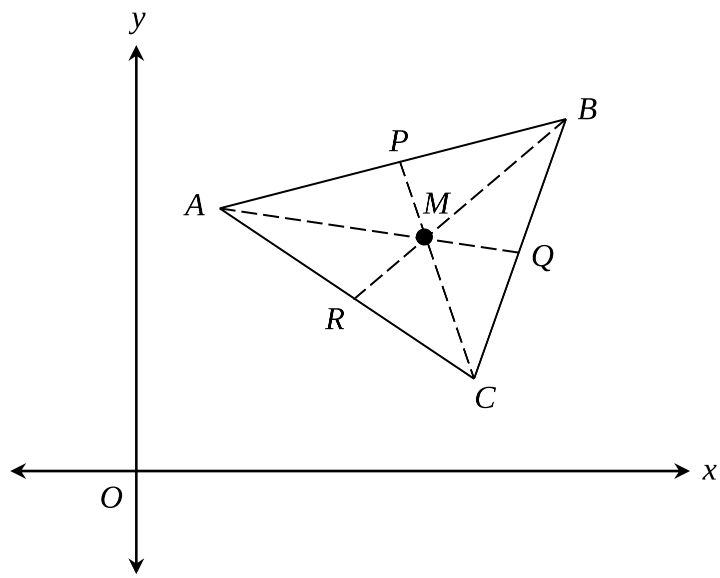
has no real roots if $k > \frac{37}{60}$.

Question 31

Find $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$. **3**

Question 32

In ΔABC, the medians of the sides AB, BC and AC drawn from their respective midpoints P, Q and R are concurrent at point M (known as the 'centroid'), where O is the origin. This is shown in the diagram below.



(a) Let $\overrightarrow{AB}=\underset{\sim}{u}$, $\overrightarrow{BC}=\underset{\sim}{v}$, $\overrightarrow{AM}=\lambda\overrightarrow{AQ}$ and $\overrightarrow{CM}=\mu\overrightarrow{CP}$. Show that: **3**

$$AM : MQ = 2 : 1.$$

(b) Using part (i) and letting $\overrightarrow{OA}=\underset{\sim}{a}$, $\overrightarrow{OB}=\underset{\sim}{b}$ and $\overrightarrow{OC}=\underset{\sim}{c}$, show that: **2**

$$\overrightarrow{OM}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right).$$

Question 33

The Fibonacci sequence of numbers, $F_1, F_2, \ldots$ is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

(a) Show that: $F_{2n+3}F_{2n+1} - \left(F_{2n+2}\right)^2 = -F_{2n+2}F_{2n} + \left(F_{2n+1}\right)^2$ **2**

(b) Prove by mathematical induction that $F_{2n+1}F_{2n-1} - \left(F_{2n}\right)^2 = 1$ for all positive integers n. **3**

(c) Hence, show that $\left(F_{2n}\right)^2 + 1$ is divisible by F_{2n+1}. **1**

(d) Prove that $\left(F_{2n-1}\right)^2 + 1$ is divisible by F_{2n+1}. **2**

End of paper.