NEXT EDUCATION

2020

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- All answers should be written on this examination paper.
- There is some extra writing space at the end of this paper.

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 27

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Circle the best solution below for Questions 1 – 10

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1 Which of the following is equivalent to the integral $\int \frac{\log_e x}{x}\, dx$?

(A) $\log_e\left(\frac{1}{x}\right) + c$

(B) $\log_e\left(\frac{\log_e x}{x}\right) + c$

(C) $-\log_e x + c$

(D) $\frac{\left(\log_e x\right)^2}{2} + c$

2 Which of the following represents the contrapositive statement to the following:

If $m + n \geq 2$, then $m \geq 1$ or $n \geq 1$, where $m, n \in \mathbb{R}$.

(A) Suppose $m + n \leq 2$, then $m \geq 1$ and $n \geq 1$.

(B) Suppose $m + n < 2$, then $m < 1$ or $n < 1$.

(C) Suppose $m \leq 1$ or $n \leq 1$, then $m + n \leq 2$.

(D) Suppose $m < 1$ and $n < 1$, then $m + n < 2$.

3 The point $P(-3,2,3)$ lies on the 3-dimension number field. Which of the following represents the unit vector parallel to $\overrightarrow{OP}$?

(A) $\frac{1}{\sqrt{22}}\left(-3\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}\right)$.

(B) $\frac{1}{\sqrt{2}}\left(-3\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}\right)$.

(C) $\frac{1}{\sqrt{22}}\left(3\underset{\sim}{i}-2\underset{\sim}{j}-3\underset{\sim}{k}\right)$.

(D) $\frac{1}{\sqrt{2}}\left(3\underset{\sim}{i}-2\underset{\sim}{j}-3\underset{\sim}{k}\right)$.

4 If $|z-5+12i|=13$, what is the maximum value of $|z|$?

(A) $\sqrt{13}$

(B) 13

(C) $\sqrt{26}$

(D) 26

5 A particle is moving in simple harmonic motion along the x-axis. Its velocity, v, in m/s at displacement x metres is given by the formula: $v^2 = 24 - 8x - 2x^2$.

The maximum speed of the particle is:

(A) 24 m/s

(B) 32 m/s

(C) $2\sqrt{6}$ m/s

(D) $4\sqrt{2}$ m/s

6 Which of the following is equivalent to the integral $\int \frac{\cos x}{\sin x + \sin^2 x}\, dx$?

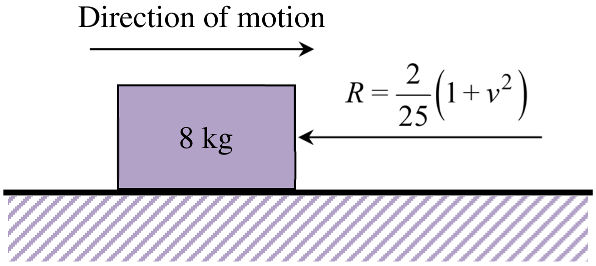
(A) $\log_e \left| \frac{\cos x}{\sin x + \sin^2 x} \right| + c$

(B) $\log_e (\cos x) + c$

(C) $\log_e \left(\sin x + \sin^2 x \right) + c$

(D) $\log_e \left| \frac{\sin x}{\sin x + 1} \right| + c$

7 An 8 kg object moving along a horizontal plane with initial velocity of 40 m/s experiences a resistive force R of $\frac{2}{25}\left(1+v^2\right)$, where v is the velocity of the object after t seconds.



Which of the following represents the velocity of the object after t seconds?

(A) $v = \dfrac{40 - \tan\left(\frac{t}{100}\right)}{1 + 40\tan\left(\frac{t}{100}\right)}$.

(B) $v = 100\log_e\left(\frac{40 - t}{40 + t}\right)$.

(C) $v = 10\sqrt{\log_e\left(4 - \frac{t}{40 + t}\right)}$.

(D) $v = 80\tan\left(1 + \frac{400 - t}{10}\right)$.

8 The solutions to the equation $x^3 - 3x^2 - 3x + 1 = 0$ are $x = \tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$ and -1.

What is the value of $\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12}$?

(A) 3

(B) 6

(C) 14

(D) 15

9 Find the point of intersection between the vectors $\underset{\sim}{r}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\underset{\sim}{r}_2 = \begin{bmatrix} -1 \\ -5 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where λ and μ are constants.

(A) (–5,2)

(B) (2,–5)

(C) (5,–2)

(D) (–2,5)

10 Use the substitution $u = \sqrt[6]{x}$ which of the following is equivalent to $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$?

(A) $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log_e\left|\sqrt[6]{x} + 1\right| + c$.

(B) $\log_e\left|\sqrt{x} + \sqrt[3]{x}\right| + c$.

(C) $\sqrt{x} + \sqrt[3]{x} + \sqrt[6]{x} + c$.

(D) $2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} + c$.

End of Section I.

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

You should include all your solutions and working in the spaces provided in this paper.

Question 11 (15 marks)

(a) If $z = 5 - 2i$ and $w = -1 + 3i$, express each of the following in the form $a + ib$, where a and b are real.

(i) $2z + w$. **1**

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(ii) $w\overline{w}$. **1**

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(iii) z^2. **1**

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Question 11 continues on the next page.

(iv) $\dfrac{w}{\overline{z}}$. **2**

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(b) The coordinates $P\left(-2+\sqrt{3},3\right)$, $Q\left(-2\ ,2\right)$ and $R\left(2\sqrt{3}-2,0\right)$ lie on the 3-dimensional number field. With the use of vectors, find the size of $\angle PQR$. **2**

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(c) Find $\displaystyle\int \frac{1}{\sqrt{1-x-x^2}}\,dx$. **2**

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Question 11 continues on the next page.

(d) A sphere has the equation $x^2 + y^2 + z^2 - 4x + 6z - 23 = 0$. **2**
Find the sphere's centre and radius.

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(e) (i) Find the real numbers a, b and c such that: **2**

$$\frac{x^2 + 3x + 14}{x^3 - 2x^2 + 4x - 8} = \frac{a}{x-2} + \frac{bx + c}{x^2 + 4}.$$

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(ii) Hence, or otherwise, find $\int \frac{x^2 + 3x + 14}{x^3 - 2x^2 + 4x - 8}\, dx$. **2**

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End of Question 11.

Question 12 (15 marks)

(a) For the two vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$, find the value of m if.

(i) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular. **1**

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(ii) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel. **1**

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(iii) The length of $\underset{\sim}{a}$ is 5 units. **1**

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(b) Express $\left(1-\sqrt{3}i\right)^{10}$ in the form $a + ib$, where a and b are real numbers. **2**

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Question 12 continues on the next page.

(c) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{1-\cos x}$. **3**

Question 12 continues on the next page.

(d) Solve for z: $\cos^2\theta z^2 + \sin 2\theta z + 1 = 0$. **3**

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(e) The complex number $z = x + iy$, where $x, y \in \mathbb{R}$, satisfies the inequality:

$$|z - 3 - 3i| \leq 2\sqrt{2}\,.$$

(i) Sketch the locus of the point P on an Argand diagram in the space below representing the complex number z, showing all key features. **2**

Question 12 continues on the next page.

(ii) Find the maximum values for $|z|$ and $\arg z$, leaving your answer in exact form. **2**

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End of Question 12.

Question 13 (15 marks)

(a) Find the equation of a vector that passes through the point $P(-2,5)$ and is perpendicular to the vector $2\underset{\sim}{i} - 3\underset{\sim}{j}$. **2**

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(b) With the use of Euler's formula find the value of i^i, rounding your solution to four significant figures. **2**

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(c) Find $\int x \tan^{-1} x \, dx$. **2**

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Question 13 continues on the next page.

(d) Solve for z, expressing your answer in the form $a + ib$, where a and b are real numbers. **2**

$$z^2 = 11 - 60i.$$

(e) The coordinates $P(-2,4,5)$ and $Q(1,-2,3)$ lie on the 3-dimensional number field.

(i) Find the equation of the vector that passes through P and Q, expressing your answer in the form $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda \underset{\sim}{d}$, where $\lambda \in \mathbb{R}$. **2**

(ii) Hence, or otherwise, find the Cartesian equation of the line passing through P and Q. **1**

Question 13 continues on the next page.

(f) The tides observed in Mar Harbour can be modelled using simple harmonic motion. On average, the first daily low tide is observed at 4:30am while the first high tide is observed at 1:00pm. Low tides are measured at 3m above sea level on average while high tides are measured at 11m above sea level.

(i) If the tides in Mar Harbour are modelled using the following formula: **2**

$$h = a - b\cos(nt)$$

where t represents time in hours after the first daily low tide, and a, b and n are constants.

Find the value of a, b and n.

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(ii) Find the first time each day when the tide is increasing at the fastest rate, rounding your answer to the nearest minute. **2**

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End of Question 13.

Question 14 (15 marks)

(a) Prove by contradiction that n is irrational given that $3^n = 2$, where $n \in \mathbb{Z}$. **2**

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(b) (i) Let $I_n = \int t^n e^{at}\, dt$ for integers $n \geq 0$, where a is a constant. **2**

Show that $I_n = \dfrac{t^n e^{at}}{a} - \dfrac{n}{a} I_{n-1}$ for integers $n \geq 1$.

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Question 14 continues on the next page.

(ii) Hence, find the value of $\int_0^1 t^3 e^{2t}\, dt$. **2**

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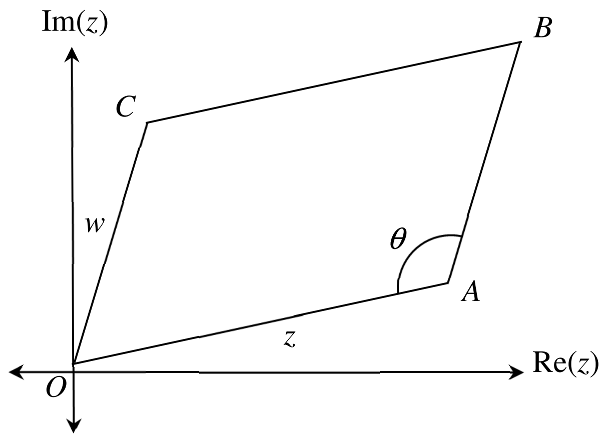
(c) On the Argand diagram $OABC$ is a parallelogram where O is the origin, $\angle OAB = \theta$, and A and C are represented by the complex numbers z and w respectively, as shown in the diagram below.



Question 14 continues on the next page.

(i) Show that $\arg\left(\dfrac{w}{z}\right) = \pi - \theta$. **1**

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(ii) Show that $|z + w|^2 = |z|^2 + |w|^2 + 2|z||w|\cos\left[\arg\left(\dfrac{w}{z}\right)\right]$. **1**

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(iii) Hence, or otherwise, show that $\cos\left[\arg\left(\dfrac{w}{z}\right)\right] = \dfrac{|z+w|^2 - |w-z|^2}{4|z||w|}$. **2**

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(iv) If $|w| = 2$, $\theta = \dfrac{7\pi}{12}$, and $z = 3\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$, find an expression for the vector $z + w$ in Cartesian form. **2**

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Question 14 continues on the next page.

(d) If $z = \mathrm{r}(\cos\theta + i\sin\theta)$, show that $\dfrac{z^2 - r^2}{z}$ is purely imaginary, and state its value in terms of r and θ. **3**

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End of Question 14.

Question 15 (15 marks)

(a) Find $\displaystyle\int \frac{18\sin x + \cos x}{2\sin x + 3\cos x}\,dx$. **3**

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(b) An object of mass m kg is released through a medium with resistance of $\dfrac{mv^2}{40}$, where v is the velocity of the object after t seconds. Assuming acceleration due to gravity of 10 m/s^2,

(i) Show that $v = \dfrac{20\left(e^t - 1\right)}{e^t + 1}$. **3**

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Question 15 continues on the next page.

(ii) Show that $x = 20\log_e\left(\dfrac{400}{400 - v^2}\right)$. **2**

(iii) Find the distance travelled by the object after 4 seconds (nearest metre). **1**

Question 15 continues on the next page.

(c) Consider the equation $z^5 - i = 0$.

(i) Show that $1 - iz - z^2 + iz^3 + z^4 = 0$, for $z \neq i$. **1**

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(ii) Hence or otherwise, find all the roots of $z^5 - i = 0$ in the $cis\theta$ form. **1**

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(iii) Show that: $(z-i)\left(z^2 - 2i\sin\dfrac{\pi}{10}z - 1\right)\left(z^2 + 2i\sin\dfrac{3\pi}{10}z - 1\right) = 0$. **2**

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Question 15 continues on the next page.

(iv) Hence, show that: $\sin\dfrac{\pi}{10}\sin\dfrac{3\pi}{10} = \dfrac{1}{4}$. **2**

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End of Question 15.

Question 16 (15 marks)

(a) By methods of mathematical induction, prove for integers $n \geq 0$: **3**

$$\cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos\left(2^n\alpha\right) = \frac{\sin\left(2^{n+1}\alpha\right)}{2^{n+1}\sin\alpha}.$$

Question 16 continues on the next page.

(b) Find $\int \frac{1}{\sqrt{e^{2x}+1}}\, dx$. **3**

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(c) If a, b and c are positive real numbers and $a + b \geq c$, prove that: **2**

$$\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} \geq 0 .$$

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Question 16 continues on the next page.

(d) A 5 kg cannonball was fired towards the sea from a cannon perched 200m above sea level on the edge of a cliff. It was fired with initial velocity of $80\sqrt{2}$ m/s at an angle of 45° to the horizon. In addition to gravity of 10 m/s^2, the sea wind exerts a resistive force proportional to the cannonball's velocity v of $\frac{v}{20}$. The horizontal and vertical equations of the cannonball's acceleration is as follows (**DO NOT PROVE THESE**):

$$\ddot{x} = -\frac{\dot{x}}{100} \qquad \ddot{y} = -10 - \frac{\dot{y}}{100}$$

(i) Prove the following horizontal and vertical equations of motion. **3**

$$\dot{x} = 80e^{-\frac{t}{100}} \qquad \dot{y} = 1080e^{-\frac{t}{100}} - 1000$$

$$x = 8000 - 8000e^{-\frac{t}{100}} \qquad y = -108000e^{-\frac{t}{100}} - 1000t + 108200$$

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Question 16 continues on the next page.

(ii) The cannonball takes approximately 17.9 seconds to hit the sea water below (**DO NOT PROVE THIS**). **2**

Verify this result, and find the horizontal distance from the cliff the cannonball hits the sea, rounding your answer to the nearest metre.

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(iii) Find the time it takes for the cannonball to reach its maximum height, rounding your answer to one decimal place. **1**

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(iv) Using the rounded solution in part (iii), find the maximum height attained by the cannonball, rounding your answer to the nearest metre. **1**

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End of paper.

Extra writing space

If you use this space, clearly indicate white question you are answering.