NExT EDUCATION

2016

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10

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1 What are the complex solutions for z in the equation $z^2 + iz + 2 = 0$?

(A) $z = -2i, i$

(B) $z = 2i, -i$

(C) $z = \dfrac{i \pm 3}{2}$

(D) $z = \dfrac{-i \pm 3}{2}$

2 Find the value of the eccentricity (e) of the following equation: $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

(A) $e = \dfrac{4}{3}$

(B) $e = \dfrac{5}{3}$

(C) $e = \dfrac{3}{5}$

(D) None of the above

3 Which expression is equal to $\int \frac{e^x}{e^{2x}+1}\,dx$?

(A) $\tan^{-1}\left(e^x\right)+c$

(B) $\tan^{-1}\left(e^{2x}+1\right)+c$

(C) $\log_e\left(e^x+1\right)+c$

(D) $\log_e\left(e^{2x}+1\right)+c$

4 The equation $x^3+5x^2+4x-1=0$ has roots $x = \alpha, \beta$ and γ.
Find the value of $\alpha^3+\beta^3+\gamma^3$.

(A) –107

(B) –92

(C) –62

(D) –32

5 In how many ways can 24 identical marbles be placed in 5 different jars?

(A) $\frac{24!}{5!}$

(B) $\frac{29!}{5!}$

(C) $\frac{28!}{24!4!}$

(D) $\frac{29!}{24!5!}$

6 The area enclosed by the circle $(x - a)^2 + y^2 = b^2$, where $a > b > 0$, is rotated about the y-axis. What is the volume of the *torus* formed?

(A) $ab\pi^2$ units^3

(B) $2ab^2\pi^2$ units^3

(C) $3a^2b^2\pi^2$ units^3

(D) $4a^2b^3\pi^2$ units^3

7 For the curve $x^2 + xy + y^2 = 9$, which of the following is a point where the tangent to the curve is a vertical line?

(A) $\left(\sqrt{3}, -2\sqrt{3}\right)$

(B) $\left(-2\sqrt{3}, \sqrt{3}\right)$

(C) $(0, 3)$

(D) $(-3, 0)$

8 A car was travelling along a circular bend with radius of 500m banked at an angle of 30°. Assuming gravity of 9.8ms^{-2}, at what velocity should the car travel such that the lateral force is eliminated?

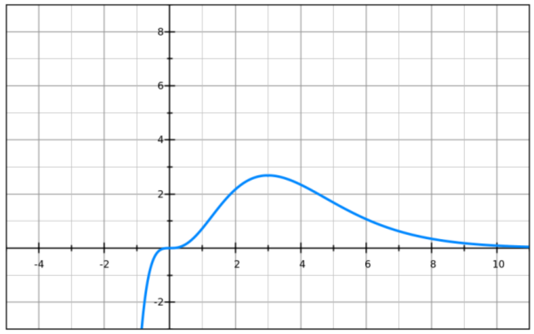
(A) 49.5 ms^{-1}

(B) 53.2 ms^{-1}

(C) 2450 ms^{-1}

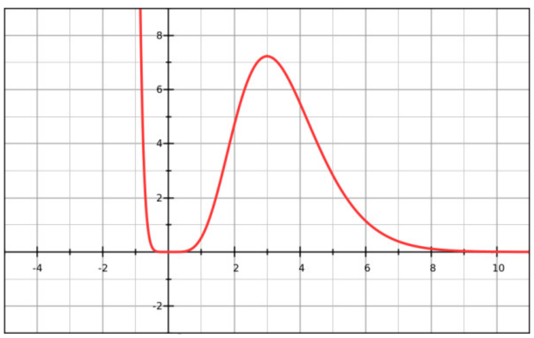
(D) 2829.0 ms^{-1}

9 The following diagram shows the graph of $y = 2x^3e^{-x}$:

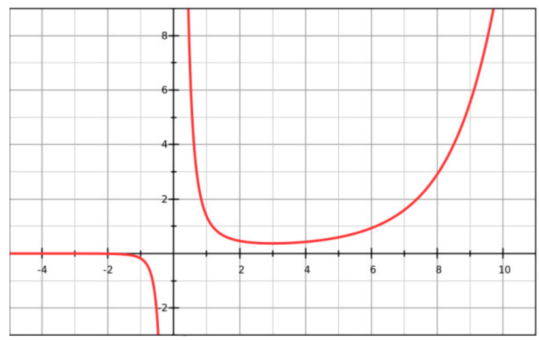


Which of the following graphs best represents $y = \sqrt{2x^3e^{-x}}$?

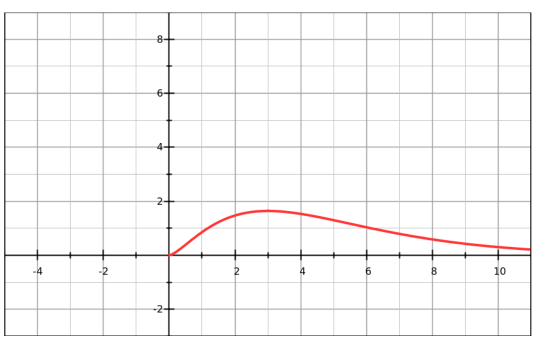
(A)



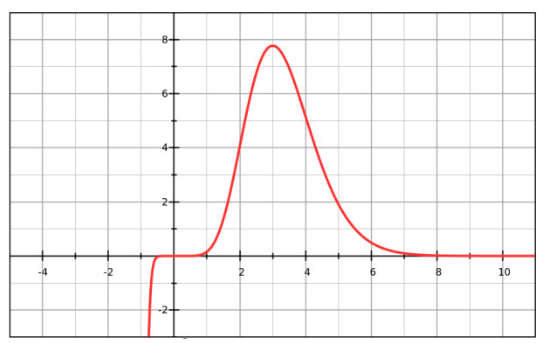
(B)



(C)



(D)



10 ω is a complex cube root of unit. Which of the following equates to

$$(1 - 3\omega + \omega^2)(1 + \omega - 8\omega^2)?$$

(A) 1

(B) 9

(C) 24

(D) 36

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question on a NEW page on your OWN PAPER.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a NEW page on your OWN PAPER.

(a) Use integration by parts to find $\int x \ln x \, dx$. **2**

(b) Let $z = 2 - 2i$.

(i) Express z in modulus-argument form. **2**

(ii) Express z^{20} in modulus-argument form. **2**

(c) (i) Find real numbers a, b and c such that **2**

$$\frac{2x^3 + 2x^2 - 18}{x^2(x+3)(x-3)} = \frac{a}{x^2} + \frac{b}{x+3} + \frac{c}{x-3}.$$

(ii) Hence, or otherwise, find $\int \frac{2x^3 + 2x^2 - 18}{x^2(x+3)(x-3)} \, dx$. **2**

(d) Sketch the following on different complex planes labelling all key features:

(i) $\text{Im}(z) = |z|$. **2**

(ii) $\text{Arg}(z-2) - \text{Arg}(z) = \frac{\pi}{3}$. **3**

Question 12 (15 marks) Use a NEW page on your OWN PAPER.

(a) Find $\int \frac{x}{x^4+1}\,dx$. **2**

(b) Let $w = -3 - 4i$.

(i) Find $w + \overline{w}$. **1**

(ii) Express $\sqrt{w}$ in the form $a + ib$, where a and b are real numbers. **2**

(iii) Using (ii), or otherwise, solve for z in the form $x + iy$: **2**

$$z^2 - 3z + (3 + i) = 0$$

(c) If α, β and γ are the roots of the equation $x^3 + 9x^2 - 4x - 8 = 0$,

(i) Find an equation with roots of α^2, β^2 and γ^2. **2**

(ii) Find the value of $\frac{\alpha}{\beta} + \frac{\alpha}{\gamma} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} + \frac{\gamma}{\beta}$. **3**

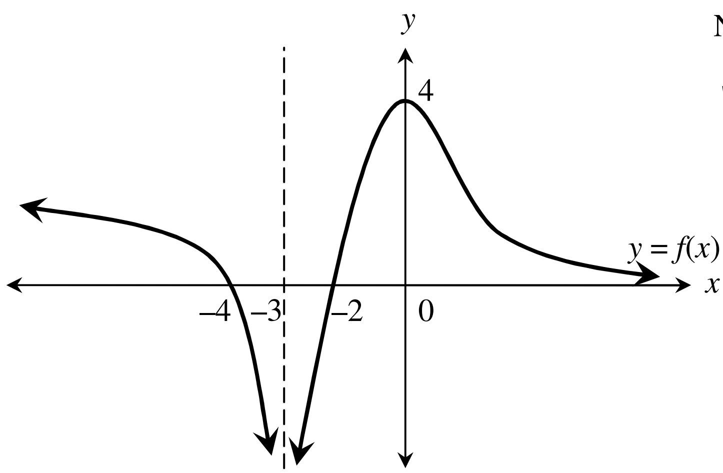
(d) Let z be a complex number such that $|z| = a$ and $\text{Arg}z = \theta$, where $0 < \theta < \frac{\pi}{2}$. **3**

Prove that $\text{Arg}\left(a^2 - z^2\right) = \theta - \frac{\pi}{2}$.

Question 13 (15 marks) Use a NEW page on your OWN PAPER.

(a) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int \frac{1}{1+\sin x - \cos x}\,dx$. **3**

(b) The diagram shows the graph of a function $f(x)$.



Sketch the following curves on separate half-page diagrams.

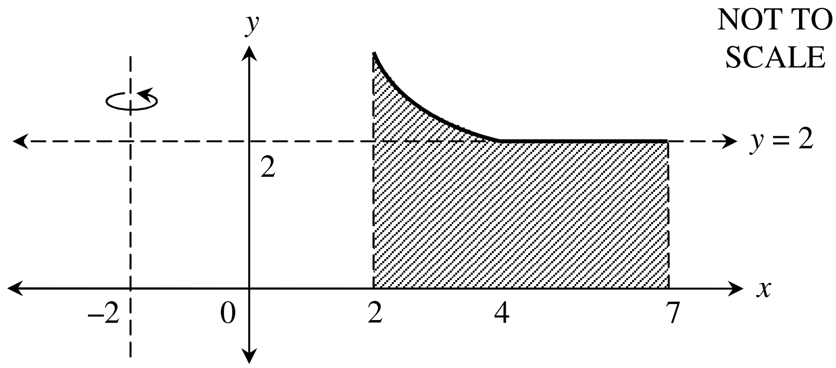
(i) $y = \left[f(x)\right]^2$ **2**

(ii) $y = f\left(|x|\right)$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \ln\left[f(x)\right]$ **2**

(c) In the diagram below, the shaded area is comprised of two regions, one bound by the graph $y = \dfrac{6}{x-1}$ and the x-axis between $x = 2$ and $x = 4$, and the other bound by the $y = 2$ and the x-axis between $x = 4$ and $x = 7$. **4**



Using the method of cylindrical shells, find the volume of the solid formed when the shaded region in the diagram is rotated about the line $x = -2$.

Question 14 (15 marks) Use a NEW page on your OWN PAPER.

(a) A particle of mass m is falling through a medium with resistance mkv^2, starting from rest. Assuming gravity of g m/s^2,

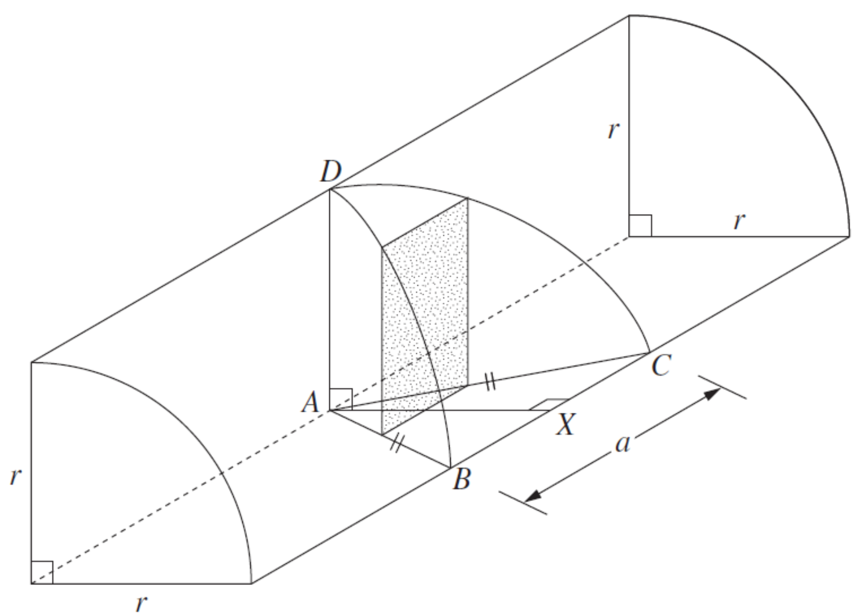
(i) Show that the particle's terminal velocity V is $\sqrt{\frac{g}{k}}$. **1**

(ii) If the particle's velocity after t seconds is v ms^{-1}, show that: **3**

$$v = V\left(\frac{e^{\frac{2gt}{V}} - 1}{e^{\frac{2gt}{V}} + 1}\right).$$

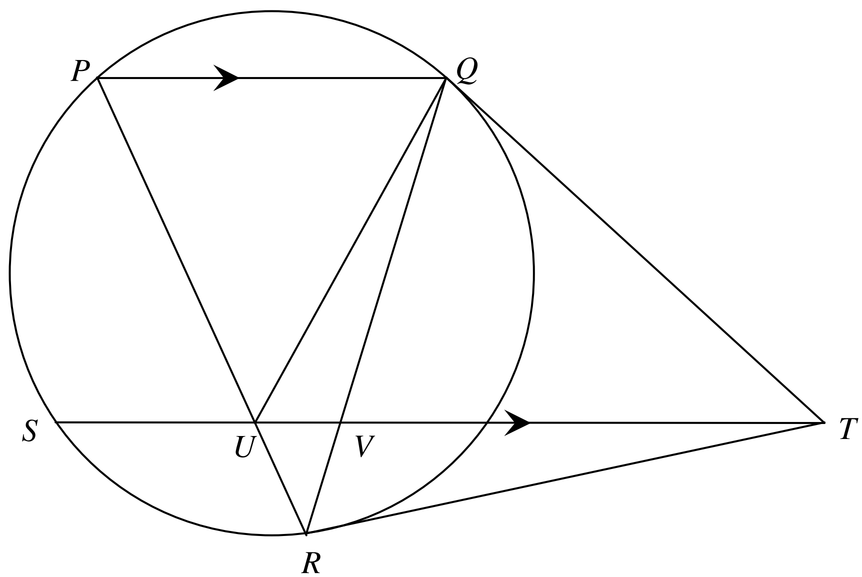
(b) The solid $ABCD$ is cut from a quarter cylinder of radius r as shown. Its base is an isosceles triangle ABC with $AB = AC$. The length of BC is a and the midpoint of BC is X. **4**

The cross-sections perpendicular to AX are rectangles. A typical cross-section is shown shaded in the diagram.



Find the volume of the solid $ABCD$.

(c) In the diagram, from an external point T two tangents are drawn to a circle meeting the circle at Q and R. A line PQ is drawn, where P lies on the circumference of the circle. Another line ST is drawn parallel to PQ, where S lies on the circumference of the circle. ST meets the lines PR and QR at U and V respectively.



Copy this diagram.

(i) Prove that ΔTVR is similar to ΔTRU. **2**

(ii) Show that $TU.TV = TQ^2$. **2**

(iii) Prove that ΔVQT is similar to ΔQUT. **2**

(iv) Show that ΔPUQ is isosceles. **1**

Question 15 (15 marks) Use a NEW page on your OWN PAPER.

(a) (i) Show that $3x^2 + 18x - 5y^2 + 10y + 7 = 0$ is the equation of a hyperbola. **1**

(ii) For the hyperbola in (i), find the eccentricity and hence the coordinates of the foci and the equation of the directrices. **2**

(b) a, b and c are the three sides of a triangle.

(i) Show that $ab + ac + bc \leq a^2 + b^2 + c^2$. **1**

(ii) Show that $3(ab + ac + bc) \leq (a + b + c)^2 \leq 4(ab + ac + bc)$. **3**

(c) (i) Let $U_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ for integers $n \geq 2$. **2**

Show that $U_n + U_{n-2} = \dfrac{1}{n-1}$.

(ii) Hence, or otherwise, evaluate: $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$. **2**

(d) Consider the equation $z^5 - i = 0$.

(i) Show that $z = i$ is a solution to the equation, and hence show that **1**

$$1 - iz - z^2 + iz^3 + z^4 = 0, \text{ for } z \neq i.$$

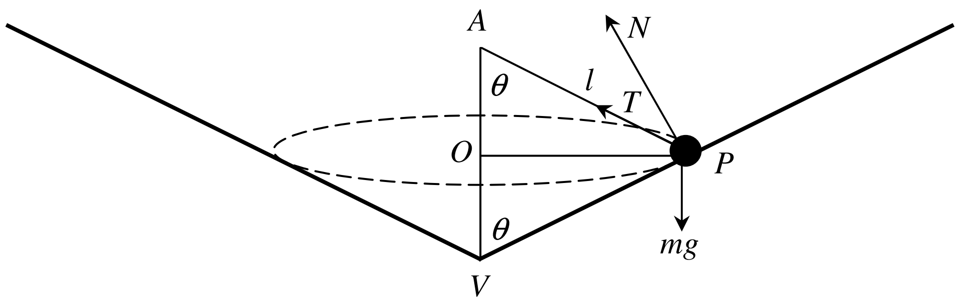
(ii) Hence or otherwise, find all the roots of $z^5 - i = 0$. **1**
[You may leave your solution in the $cis\theta$ form.]

(iii) Show that $(z - i)\left(z^2 - 2i\sin\frac{\pi}{10}z - 1\right)\left(z^2 + 2i\sin\frac{3\pi}{10}z - 1\right) = 0$. **2**

Question 16 (15 marks) Use a NEW page on your OWN PAPER.

(a) Solve for x in general form: $\sin 2x + \sin 3x + \sin 4x = 0$. **3**

(b) An object P of mass m kg is connected to a fixed point A by a light, inextensible string with length l vertically above the vertex of a cone.



The object makes an angle θ, with the vertical AV where $0° < \theta < 45°$ and moves in a circular motion on the horizontal plane. The object moves with constant angular velocity ω radians per second around O, the centre of the circle.

The tension in the string is T Newtons and the normal reaction force from the cone onto the particle is N Newtons.

(i) Draw a diagram showing the forces on P. **2**

Show that:

$$T = \frac{m}{\cos 2\theta}\left(g\cos\theta - \omega^2 l \sin^2\theta\right)$$

$$N = \frac{m\sin\theta}{\cos 2\theta}\left(\omega^2 l\cos\theta - g\right).$$

(ii) Hence, or otherwise, show that the condition for the object P to remain in contact with the surface of the cone is $\omega > \sqrt{\dfrac{g}{l\cos\theta}}$. **2**

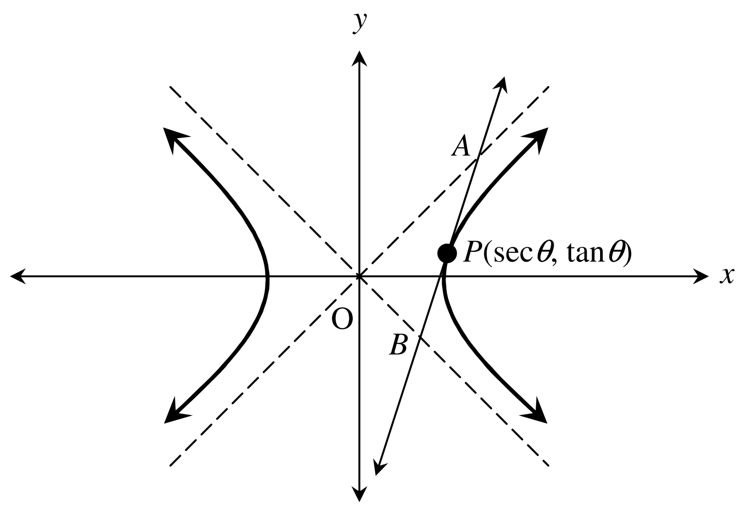
(c) A sequence of numbers u_n is defined as follows: **3**

$$\begin{cases} u_1 = u_2 = 1 \\ u_{n+1} = u_n + u_{n-1}, \text{ for } n \geq 2. \end{cases}$$

By using mathematical induction, show that for $n \geq 1$: $u_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$,

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of $x^2 - x - 1 = 0$.

(d) $P(\sec\theta, \tan\theta)$ is a point that lies on the hyperbola $x^2 - y^2 = 1$, as shown in the diagram below.



The tangent to the hyperbola at P meets the asymptotes $y = \pm x$ at A and B.

(i) Show that the equation of the tangent at P is $x\sec\theta - y\tan\theta = 1$. **1**

(ii) Show that $AP = PB$. **2**

(iii) Show that the area of ΔOAB is independent of the position of P. **2**

End of paper.